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TEMPORAL AGGREGATION AND RELATED PROBLEMS IN MULTIVARIATE TIME SERIES ANALYSIS

A Dissertation

Submitted to

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DOCTOR OF PHILOSOPHY

by

Ceylan Yozgatligil

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ABSTRACT

TEMPORAL AGGREGATION AND RELATED PROBLEMS IN MULTIVARIATE TIME SERIES ANALYSIS

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The time series data used are generally sums over time of data generated more frequently than the reporting interval. In this research, we focused on the effect of temporal aggregation on a vector autoregressive moving average (VARMA) model structure, a cointegration relationship, the causality, and multiplicative seasonal VARMA processes.

First, we worked on the cointegration problem and showed that while the cointegrating matrix remains unchanged, temporal aggregation changes the model form and affects the results of the cointegration trace test. We derived a modified test statistic and proved that the limiting distribution of the new statistic is the same as that of Johansen's trace test statistic. We can use Johansen's table of critical values but we have to use the modified test statistic that incorporates the effect of aggregation in computing

the test statistic when aggregate data are used for the test.

The use of aggregate data for causal inference is common in practice. Since the form of the vector time series model changes after aggregation, non-causality conditions for the basic model and for the model of aggregates are different. Temporal aggregation often deduces a causal relationship between aggregate variables. Because the standard test fails to detect cointegration in aggregate series, we developed a modified testing procedure to test the Granger non-causality in cointegrated systems for aggregates.

Many business and economic time series show seasonality. The best way to present seasonality is by using multiplicative models. We studied the representation problem in multiplicative seasonal VARMA models and showed that the correct order of non-seasonal and seasonal parameters in the representation improves parameter estimation and forecasts. We recommend fitting a multiplicative model by using different representations and making selection with information criteria. We also derived the model for aggregates of multiplicative processes.

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CHAPTER 1

INTRODUCTION AND LITERATURE SURVEY

1.1 Introduction

Economic theory states that certain pairs of economic variables should not drift too far apart, at least in the long-run. If they diverge too much, other forces such as market mechanisms or government intervention will bring them back together. The equilibrium relationships have a similar behavior. If $\boldsymbol{x}_t$ is a vector of time series variables, then equilibrium occurs when the specific constraint $z_t = \boldsymbol{\beta}'\boldsymbol{x}_t$ is stationary. Although individual variables in $\boldsymbol{x}_t$ are not be stationary, if there is equilibrium, then these variables will move together and z_t becomes stable. In other words, each individual variable is integrated, but a linear combination of these component variables is stationary. Integrated component variables with this property are said to be cointegrated.

One of the most important goals of empirical research is to explain the causal relationship among a set of variables such as money-income, revenue-expenditure, inflation-growth, and so on. It has been recognized that the existence of a high correlation among variates does not imply that they are causally related (Pierce and Haugh, 1977). The most well-known method to measure the causal relationship between variables is the Granger causality which has been introduced by Granger (1969). This concept is defined in terms of predictability and exploits the direction of the flow of time to achieve a causal ordering of associated variables. It is appropriate for empirical model building strategies because it does not rely on the specification of a statistical model.

Multivariate vector time series processes are very useful for studying the relationship between several variables. Many business and economic time series have a seasonal behavior, which means that they show a recurrence of some recognizable pattern after some regular interval (called the seasonal period and denoted by s). Some applied researchers prefer to use officially adjusted time series; however, many studies emphasize the importance of using unadjusted time series. Most notably, Sims (1974), Wallis (1974) and Ghysels (1988) claim that the official seasonal adjustment results create dynamic, biased relationships and provide a weak relationship between seasonally adjusted series of production, sales, and inventories. Because of these and other drawbacks of seasonally adjusted series, multiplicative vector time series models for unadjusted seasonal series are introduced in the literature.

In time series analysis, data are often available in the form of a temporal aggregation or a systematical sampling. As a result, the analysis, modeling, and testing of a vector time series is frequently based on aggregated data.

The purpose of this research is to analyze the effects of temporal aggregation through the testing of a multivariate time series for cointegration and causality in cointegrated systems. We will investigate the validity and effectiveness of various causality and cointegration tests when aggregate time series are used. We will also study the representation of multiplicative autoregressive moving average models. The consequences of different representations on parameter estimation, forecasting, and causality will be examined carefully. From these analyzes, we will determine the best summary statistic to represent a multiplicative vector process.

The relevant literature will be reviewed first in Chapter 1, which includes the presentation of problems. Chapter 2 is the derivation of the multivariate autoregressive moving average processes of aggregate data. Chapter 3 investigates the testing procedure for cointegration under temporal aggregation. Chapter 4 studies the effects of temporal aggregation on the Granger causality and the causality tests in cointegrated systems. Chapter 5 examines the problem of representing multiplicative vector autoregressive moving average models. Finally, Chapter 6 gives the conclusion and offers further topics for future research.

1.2 Vector Autoregressive Moving Average Models

Multivariate vector time series processes are very useful for studying the relationship between several variables. The vector autoregressive moving average (VARMA) models are used to represent vector time series. In this section, we will introduce widely used VARMA models in this study.

Let $\{\boldsymbol{x}_t\}$, $t = 0, \pm 1, \pm 2, \dots$, with $\boldsymbol{x}_t = (\mathrm{x}_{1t} \quad \mathrm{x}_{2t} \quad \cdots \quad \mathrm{x}_{kt})'$ be a zero mean, covariance stationary, purely nondeterministic, k-dimensional vector time series. It is assumed that $\{\boldsymbol{x}_t\}$ admits the vector autoregressive-moving average VARMA(p,q) process

$$\boldsymbol{\phi}_{\mathrm{p}}(\mathrm{B}) \underset{\mathrm{k}\times 1}{\boldsymbol{x}_t} = \boldsymbol{\theta}_{\mathrm{q}}(\mathrm{B}) \underset{\mathrm{k}\times 1}{\boldsymbol{a}_t} \tag{1.1}$$

where $\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\phi}_1\mathrm{B} - \dots - \boldsymbol{\phi}_{\mathrm{p}}\mathrm{B}^{\mathrm{p}}$ and $\boldsymbol{\theta}_{\mathrm{q}}(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\theta}_1\mathrm{B} - \dots - \boldsymbol{\theta}_{\mathrm{q}}\mathrm{B}^{\mathrm{q}}$ are matrix polynomials in the backshift operator B, defined by $\mathrm{B}^{\mathrm{j}}\boldsymbol{V}_t = \boldsymbol{V}_{t-j}$ for any integer j and vector $\boldsymbol{V}_t$. $\boldsymbol{I}_k$ is the k-dimensional identity matrix, the $\boldsymbol{\phi}$'s and $\boldsymbol{\theta}$'s are $k \times k$ parameter matrices, and $\boldsymbol{a}_t$ is a zero mean white noise vector with $\mathrm{E}(\boldsymbol{a}_t\boldsymbol{a}_t') = \boldsymbol{\Omega}$ and $\mathrm{E}(\boldsymbol{a}_t\boldsymbol{a}_{t+s}') = \boldsymbol{0}$ for $s \neq 0$. When p=0, the $VARMA(0,q)$ model is referred to as a vector moving average model of order q; $VMA(q)$, and when q=0, $VARMA(p,0)$ model is referred to as a vector autoregressive model of order p; $VAR(p)$. We also assume that the zeroes of the determinantal polynomial $|\boldsymbol{\phi}_p(B)|$ are all outside the unit circle so that $\boldsymbol{x}_t$ is second order stationary and can be expressed as

$$\boldsymbol{x}_t = \sum_{j=0}^{\infty} \Psi_j \boldsymbol{a}_{t-j} = \Psi(\mathrm{B})\boldsymbol{a}_t \tag{1.2}$$

where $\Psi(B) = \boldsymbol{\phi}_p(B)^{-1}\boldsymbol{\theta}_q(B) = \sum_{j=0}^{\infty} \Psi_j B^j$ with $\Psi_0 = \boldsymbol{I}$ and $\sum_{j=0}^{\infty} j|\Psi_j| < \infty$. Furthermore, we assume that the zeroes of the determinantal polynomial $|\boldsymbol{\theta}_q(B)|$ are all outside the unit circle so that $\boldsymbol{x}_t$ is invertible and can be expressed as

$$\Pi(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t \tag{1.3}$$

where $\Pi(B) = \boldsymbol{\theta}_q(B)^{-1}\boldsymbol{\phi}_p(B) = -\sum_{j=0}^{\infty} \Pi_j B^j$ with $\Pi_0 = -\boldsymbol{I}$ and $\sum_{j=0}^{\infty} |\Pi_j| < \infty$.

1.3 Temporal Aggregation and Times Series Models

A time series variable is either a flow variable or a stock variable. A flow variable can be obtained through aggregation over equal time intervals. The elements of the aggregate process of a flow variable are often partial sums of the basic series; that is, $\boldsymbol{X} = \{\boldsymbol{X}_T\}_{T=0}^{\infty} = \left\{\sum_{i=0}^{m-1} \boldsymbol{x}_{mT-i}\right\}_{T=0}^{\infty}$. Stock variables are obtained by systematic sampling;

therefore, only the m^{th} elements of the original process are observed. Thus, the aggregate process for a stock variable is $X = \{X_T\}_{T=0}^{\infty} = \{x_{mT}\}_{T=0}^{\infty}$. For this study, we will concentrate on flow variables. Let x_t be the equally spaced basic series and assume that the observed time series X_T is the m-period non-overlapping aggregates of x_t defined as

$$X_T = \sum_{j=0}^{m-1} x_{mT-j} = (1+B+...+B^{m-1})x_{mT}, \tag{1.4}$$

where T is the aggregate time unit and m is in fixed order of aggregation. Thus, X_T is called an aggregated series and x_t is called a basic series. For example, if x_t is a monthly series and m is 3, then the X_T are the quarterly sum of the monthly series x_t. The number of observations of X_T is $N = [n/m]$, where n is the sample size of the basic series and m is the aggregation period. It is assumed that the aggregated series X_T is available beginning with $T = 1$.

The literature of aggregation started with Quenoville (1958) for a stationary univariate series. Quenoville discussed the effects of aggregation on ARMA (p, q) mixed models and he dealt with the case when q < p. Amemiya and Wu (1972) dealt with pure moving average models. They gave the MA order, q, of the mixed model for the aggregates to be equal to $[p+1-(p+1)/m]$, where $[\bullet]$ denoted the integer part of $\bullet$. Furthermore, they proved that the MA polynomial for the aggregate data is invertible. If the data are sums for non-overlapping intervals, Telser (1967, 1976) showed the least squares estimates of the autoregressive coefficients are inconsistent. In his paper, the relationship between the basic and aggregated autocovariances were derived for the first order autoregression with aggregation order m=2. Telser reports that a purely autoregressive model (AR(p)) transforms into a mixed model of the form ARMA (p, q) where the roots of the aggregated models AR polynomial are equal to the *m*-th power of those of the basic series model. Brewer (1973) also worked on the effect of aggregation on the general ARMA (p, q) model. He found some aspects of Quenoville's results to be misleading, which included Quenoville's discussion of the possibility for a pure autoregressive model to remain pure autoregressive after aggregation. Brewer showed that the ARMA (p, q) process would be transformed to ARMA (p, r), where r is equal to $[p+1+(q-p-1)/m]$. Those values are the maximum values of the ARMA orders. Tiao and Wei (1976) investigated the effect of temporal aggregation on the dynamic relationship between two distinct time series variables. Given the dynamic model

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} v(B)\lambda(B) & \mu(B) \\ \lambda(B) & 0 \end{bmatrix} \begin{bmatrix} a_t \\ e_t \end{bmatrix}$$

where $a_t \sim N(0,\sigma_a^2)$ and $e_t \sim N(0,\sigma_e^2)$ are independent, for the basic series, they obtained the model and its characteristics for the aggregated series. They pointed out that in the estimation of the parameters in the basic dynamic model, it is better to use the basic series if they are available because a one-sided relation transformed into a feedback system. Their results are an extension of Amemiya and Wu in the way they show that when the time series is generated from the AR(p) process, an aggregated series follows a stationary and invertible ARMA time series. The first researcher to deal with the effects of aggregation on a non-stationary ARIMA model was Tiao (1972). Tiao examined the effects of aggregation on the integrated moving average processes, IMA (d, q). He found that for any q, when *m* becomes larger, an aggregate time series of this type will be closer to the IMA (d, d) model. Tiao extended this result to the general ARIMA (p, d, q) processes. Tiao and Wei (1976) investigated the effects of temporal aggregation on the dynamic relationship between two distinct time series variables. They are the first researchers to represent the fact that the non-causal relationship turns out to be a causal one after temporal aggregation of flow variables. The effects of aggregation on the univariate, multiplicative seasonal ARIMA $(P, D, Q)_s(p, d, q)$ models was studied by Wei (1978a). Wei's results show that the ARIMA $(P, D, Q)_s(p, d, q)$ model was transformed into the ARIMA $(P, D, Q)_{s/m}(p, d, r)$ model, where the coefficients of the seasonal part of the model do not change and where r is equal to $[p+d+1+(q-p-d-1)/m]$. Wei (1978b) reveals the effects of temporal aggregation on parameter estimation in a finite distributed lag model through the least squares procedure. Lütkepohl (1986) demonstrates that if the basic series has a vector ARMA representation, the aggregate series has also a vector ARMA representation. However, Lütkepohl did not really solve the aggregate model in terms of aggregate variables, nor did he offer an explicit model form for the aggregates. Drost and Nijman (1993) have derived ARMA models with symmetric GARCH errors using aggregate data, while Mamingi (1996) deals with the impact of temporal aggregation over time on the Granger causality in error correction models through the Monte Carlo simulation technique. Marcellino (1999) derives an aggregate process of stock variables when the basic series follows a vector ARIMA process, and he studied the effect of temporal aggregation on a set of characteristics such as causality. Swanson and Breitung (2002) investigate the effect of temporal aggregation on the Granger causal relations in VAR models by using large aggregation intervals. They outline the various conditions based on the informational content of error covariance matrices and the causal structure of VAR. In the following study, we will derive models for an aggregate series of flow variables.

1.4 Cointegration

1.4.1 Introduction

Consider a vector series $\boldsymbol{x}_t$ measured at equally spaced time intervals. A univariate, non-stationary process or series $\boldsymbol{x}_t$ is said to be integrated of the order *d*, denoted as *I*(*d*), if its $(d - 1)^{th}$ difference is non-stationary but its d^{th} difference $\Delta^d \boldsymbol{x}_t = (1-\mathrm{B})^d \boldsymbol{x}_t$, is stationary. A *k*-dimensional vector time series $\boldsymbol{x}_t$ is said to be cointegrated of the order *d*, *b*, denoted as $\boldsymbol{x}_t \sim CI(d,b)$, if all the components of $\boldsymbol{x}_t$ are *I(d)*, and there exists a non-zero vector $\boldsymbol{\beta}$ such that $\boldsymbol{\beta}'\boldsymbol{x}_t \sim I(d-b)$, and $b>0$. The vector $\boldsymbol{\beta}$ is called a cointegrating vector (Engle and Granger, 1987). The most widely used d and b values are $d=1$ and $b=1$. In this case $z_t = \boldsymbol{\beta}'\boldsymbol{x}_t \sim \mathrm{I}(0)$, which is a stationary process. If there are more than two variables contained in $\boldsymbol{x}_t$, then there may be more than one, such as, *h* linearly independent cointegrating vectors. If we let $\boldsymbol{A}'$ be a vector composed of *h* linearly independent cointegrating vectors, then $\boldsymbol{A}'\boldsymbol{x}_t$ will be $I(d-b)$. When $d=1$ and $b=1$, $\boldsymbol{A}'\boldsymbol{x}_t$ is a stationary ($h \times 1$) vector where

$$\underset{\mathrm{h\times k}}{\boldsymbol{A}'} = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \cdots & \mathrm{a}_\mathrm{h} \end{bmatrix}'.$$

If each component of $\boldsymbol{x}_t$ is I(1), then there would always exist a Wold representation

$$(1-\mathrm{B})\boldsymbol{x}_t = \Psi(\mathrm{B})\boldsymbol{a}_t, \tag{1.5}$$

where $\Psi(B)=\sum_{i=0}^{\infty}\Psi_i B^i=\Psi(1)+(1-B)\Psi^*(B)$ with $\Psi^*(B)=\left[\sum_{i=1}^{\infty}-\left\{\sum_{j=0}^{i-1}B^j\right\}\Psi_i\right]$,

$\Psi(0)=\boldsymbol{I}$, and $\{r\Psi_r\}_{r=0}^{\infty}$ is absolutely summable; i.e., $\sum_{r=0}^{\infty}r|\Psi_{r,ij}|<\infty$, $\Psi_{r,ij}$ is the i^th^ row and the j^th^ column element of Ψ_r, and the $\boldsymbol{a}_t$'s are zero mean white noise vectors with the covariance matrix $\boldsymbol{\Omega}$. The above difference equation implies that

$$\boldsymbol{x}_t=\boldsymbol{x}_0+\Psi(1)(\boldsymbol{a}_1+\boldsymbol{a}_2+\ldots+\boldsymbol{a}_t)+\boldsymbol{\eta}_t-\boldsymbol{\eta}_0, \tag{1.6}$$

where $\Psi(1)=I+\Psi_1+\Psi_2+\ldots$ and $\boldsymbol{\eta}_t=\sum_{s=0}^{\infty}-\{\Psi_{s+1}+\Psi_{s+2}+\ldots\}\boldsymbol{a}_{t-s}=\sum_{s=0}^{\infty}-\boldsymbol{\alpha}_s\boldsymbol{a}_{t-s}$ with $\boldsymbol{\alpha}_s=\{\Psi_{s+1}+\Psi_{s+2}+\ldots\}$.

The concept of cointegration is first introduced by Granger in 1981. Then, the following result was obtained by Engle and Granger (1987) and came to be known as the Granger representation theorem: if $\underset{k\times 1}{\boldsymbol{x}_t}\sim \mathrm{CI}(1,1)$ with cointegrating rank h, then

i) $\Psi(1)$ is of rank $k-h$.

ii) There exists a vector ARMA representation

$$\boldsymbol{\phi}(\mathrm{B})\boldsymbol{x}_t=\boldsymbol{\theta}(\mathrm{B})\boldsymbol{a}_t, \tag{1.7}$$

where $\boldsymbol{\phi}(1)$ has rank h, $\boldsymbol{\phi}(B)=Adj(\Psi(B))/(1-B)^{h-1}$, $Adj(\Psi(B))$ is the adjoint matrix of $\Psi(B)$ in (1.5), and $\theta(B)=\det(\Psi(B))/(1-B)^h$ is a scalar lag polynomial.

iii) There exists $k\times h$ matrices $\boldsymbol{A}$ and γ, both of which have rank h, that

$$\boldsymbol{A}'\,\Psi(1)=0, \tag{1.8}$$

$$\boldsymbol{\phi}(1)=-\boldsymbol{\gamma}\boldsymbol{A}'. \tag{1.9}$$

iv) If the non-stationary process $\boldsymbol{x}_t$ has the following ARMA representation

$$\boldsymbol{\phi}(\mathrm{B})\boldsymbol{x}_t = \theta(\mathrm{B})\boldsymbol{a}_t, \tag{1.10}$$

where $\boldsymbol{\phi}(\mathrm{B}) = \boldsymbol{I} + \boldsymbol{\phi}_1\mathrm{B} + ... + \boldsymbol{\phi}_p\mathrm{B}^p$ and $\theta(B) = 1 + \theta_1 B + ... + \theta_q B^q$, then (1.10) can also be written in the following error correction representation

$$\boldsymbol{\phi}^*(\mathrm{B})\Delta\boldsymbol{x}_t = \gamma z_{t-1} + \theta(\mathrm{B})\boldsymbol{a}_t$$

where $\Delta\boldsymbol{x}_t = (1\text{-B})\boldsymbol{x}_t$, $z_{t-1} = \boldsymbol{A}'\boldsymbol{x}_{t-1}$, $\boldsymbol{\phi}^*(B) = \boldsymbol{\phi}(1) + \tilde{\boldsymbol{\phi}}(B)$, $\boldsymbol{\phi}(1) = (\boldsymbol{I} - \boldsymbol{\phi}_1 - ... - \boldsymbol{\phi}_p)$, and $\tilde{\boldsymbol{\phi}}(B) = \boldsymbol{\phi}_1 + (1+B)\boldsymbol{\phi}_2 + ... + (1 + B + ... + B^{p-1})\boldsymbol{\phi}_p$. That is, error correction representation can be given by

$$\left(\boldsymbol{\phi}(1) + \tilde{\boldsymbol{\phi}}(\boldsymbol{B})\right)\Delta\boldsymbol{x}_t = \gamma\boldsymbol{A}'\boldsymbol{x}_{t-1} + \theta(\mathrm{B})\boldsymbol{a}_t$$

or

$$\Delta\boldsymbol{x}_t = \gamma\boldsymbol{A}'\boldsymbol{x}_{t-1} + \sum_{i=1}^{p}\boldsymbol{\lambda}_i\Delta\boldsymbol{x}_{t-i} + \theta(\mathrm{B})\boldsymbol{a}_t \tag{1.11}$$

where $\gamma\boldsymbol{A}' = -\boldsymbol{I} + \boldsymbol{\phi}_1 + \cdots + \boldsymbol{\phi}_p$ and $\boldsymbol{\lambda}_i = \sum_{j=i+1}^{p}\boldsymbol{\phi}_j$. Equation (1.11) can be expanded as

$$\Delta\boldsymbol{x}_t = \gamma\boldsymbol{A}'\boldsymbol{x}_{t-1} + \boldsymbol{\lambda}_1\boldsymbol{x}_{t-1} - \boldsymbol{\lambda}_1\boldsymbol{x}_{t-2} + \boldsymbol{\lambda}_2\boldsymbol{x}_{t-2} - \boldsymbol{\lambda}_2\boldsymbol{x}_{t-3} + \cdots + \boldsymbol{\lambda}_{p-1}\boldsymbol{x}_{t-p+1} - \boldsymbol{\lambda}_{p-1}\boldsymbol{x}_{t-p} + \theta(\mathrm{B})\boldsymbol{a}_t$$

or

$$\Delta\boldsymbol{x}_t = \left(\gamma\boldsymbol{A}' + \boldsymbol{\lambda}_1\right)\boldsymbol{x}_{t-1} + \left(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1\right)\boldsymbol{x}_{t-2} + \cdots + \left(\boldsymbol{\lambda}_{p-1} - \boldsymbol{\lambda}_{p-2}\right)\boldsymbol{x}_{t-p+1} - \boldsymbol{\lambda}_{p-1}\boldsymbol{x}_{t-p} + \theta(\mathrm{B})\boldsymbol{a}_t.$$

When we add and subtract $\left(\gamma\boldsymbol{A}' + \boldsymbol{\lambda}_1\right)\boldsymbol{x}_{t-2}$ to the right side of the equation, we have

$$\Delta\boldsymbol{x}_t = \left(\gamma\boldsymbol{A}' + \boldsymbol{\lambda}_1\right)\Delta\boldsymbol{x}_{t-1} + \left(\gamma\boldsymbol{A}' + \boldsymbol{\lambda}_2\right)\boldsymbol{x}_{t-2} + \cdots + \left(\boldsymbol{\lambda}_{p-1} - \boldsymbol{\lambda}_{p-2}\right)\boldsymbol{x}_{t-p+1} - \boldsymbol{\lambda}_{p-1}\boldsymbol{x}_{t-p} + \theta(\mathrm{B})\boldsymbol{a}_t.$$

We repeat this procedure for the other lags. As a result we obtain the alternative error correction representation considers the error correction term at lag $t-p$ and is given by

$$\Delta x_t = (\gamma A' + \lambda_1)\Delta x_{t-1} + (\gamma A' + \lambda_2)\Delta x_{t-2} + \cdots + (\gamma A' + \lambda_{p-1}) x_{t-p+1} + \gamma A' x_{t-p} + \theta(B) a_t$$

$$\Delta x_t = \sum_{j=1}^{p-1} \Gamma_j \Delta x_{t-j} + \Pi x_{t-p} + a_t \quad (1.12)$$

where $\Gamma_j = \gamma A' + \lambda_j = -I + \phi_1 + \cdots + \phi_j$, j = 1,⋯, p-1 are $k \times k$, $\Pi = \gamma A' = -I + \phi_1 + \cdots + \phi_p = -\phi(1)$, A and γ are $k \times h$ parameter matrices.

The concept of cointegration was further developed by Engle and Yoo (1987, 1991), Phillips and Ouliaris (1990), Stock and Watson (1988), Phillips (1991), Johansen (1988, 1991, 1994), Lütkepohl and Claessen (1997), and Lütkepohl and Saikkonen (2000), among others.

1.4.2 Cointegration Tests

1.4.2.1 Residual Based Tests

Residual based tests rely on the residuals calculated from regressions among the levels of time series. They are designed to test the null hypothesis of no cointegration by testing that there is a unit root in the residuals against the root that is less than unity. If the null of a unit root is rejected, then the null of no cointegration is also rejected. Based on residuals of cointegrating regressions, the tests are the most widely used among empirical researchers because they are easy to use.

Working within the context of a bivariate system and with at most, one cointegrating vector, Engle and Granger (1987) proposed to estimate the cointegrating vector $\beta = (1, \beta_1)'$ by regressing the first component $x_{1,t}$ of x_t on the second $x_{2,t}$, using ordinary least squares (OLS) (called the cointegrating regression), and then testing whether the OLS residuals of this regression have a unit root, using the Augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1979). However, since the ADF test is conducted on estimated residuals, the tables of the critical values of this test in Fuller (1976) no longer apply. The correct critical values involved can be found in Engle and Yoo (1987). Consider the linear cointegrating regressions:

$$x_{1t} = c + \beta_1' x_{2t} + u_t, \tag{1.13}$$

where c is a constant and β_1 is a regression coefficient and u_t are the errors of the cointegrating regression.

The Dickey-Fuller (DF) test is a test based on the residuals of a cointegrating regression. After residuals are obtained, we consider the following auxiliary regression:

$$\Delta\hat{u}_t = \rho\hat{u}_{t-1} + \varepsilon_t, \tag{1.14}$$

where $\Delta u_t = u_t - u_{t-1}$, $u_t = y_t - \hat{y}_t$, $\hat{y}_t = \hat{c} + \hat{\beta}_1' z_t$, $\hat{c}$ is an estimated constant and $\hat{\beta}_1$ is an estimated regression coefficient, ρ is the auxiliary regression coefficient, and the ε_t are errors. If $\rho = 0$, then the residuals will be non-stationary. This means that there is no cointegration.

The Augmented Dickey Fuller (ADF) test allows for more dynamics in the regression in (1.14). That is, the test is performed using the following higher order AR model:

$$\Delta\hat{u}_t = \rho\hat{u}_{t-1} + \eta_1\Delta\hat{u}_{t-1} + \cdots + \eta_p\Delta\hat{u}_{t-p} + \varepsilon_t. \tag{1.15}$$

For the Phillips' (1987) Z_α test, use the regression, $\hat{u}_t = \alpha\hat{u}_{t-1} + v_t$ and compute the transformation of the standardized estimator $n(\hat{\alpha}-1)$ as

$$Z_\alpha = n(\hat{\alpha}-1)-(1/2)(s_{n\ell}^2 - s_v^2)\left[n^{-2}\sum_{t=2}^{n}\hat{u}_{t-1}^2\right]^{-1}, \tag{1.16}$$

where

$$s_v^2 = n^{-1}\sum_{t=1}^{n}\hat{v}_t^2, \tag{1.17}$$

and

$$s_{n\ell}^2 = n^{-1}\sum_{t=1}^{n}\hat{v}_t^2 + 2n^{-1}\sum_{s=1}^{\ell} w_{sl}\sum_{t=s+1}^{n}\hat{v}_t\hat{v}_{t-s}, \tag{1.18}$$

for some choice of lag window such as $w_{s\ell} = 1 - s/(\ell+1)$, where ℓ is the lag truncation number. Phillips (1987) suggests using a small value of ℓ because the sample autocorrelations of the first differenced time series usually decay quickly; it tests the random walk hypothesis, which allows both dependent and heterogeneous error sequences.

For the Phillips' (1987) Z_t test, consider the regression: $\hat{u}_t = \alpha\hat{u}_{t-1} + v_t$ and compute the transformation of the conventional regression t statistic, $t_\alpha = \left(\sum_{t=1}^{n} u_{t-1}^2\right)^{1/2}(\hat{\alpha}-1)\Big/\sum_{t=1}^{n} n^{-1}(u_t - \hat{\alpha}u_{t-1})^2$ as

$$Z_t = \left(\sum_{t=2}^{n}\hat{u}_{t-1}^2\right)^{1/2}(\hat{\alpha}-1)/s_{n\ell} - (1/2)(s_{n\ell}^2 - s_v^2)\left[s_{n\ell}\left(n^{-2}\sum_{t=2}^{n}\hat{u}_{t-1}^2\right)^{1/2}\right]^{-1}, \tag{1.19}$$

where $s_{n\ell}$ and s_v^2 as in (1.17) and (1.18). Also being tested is the random walk hypothesis that allows both dependent and heterogeneous error sequences.

Phillips and Ouliaris (1990) demonstrate that ADF and Z_t have the same limiting distribution. Note that the ADF test is basically a t test in a long autoregression involving the residuals $\hat{u}_t$. Based on the results of Phillips and Perron (1988) the Z_α test is more powerful when compared with Z_t and ADF for the models with positive serial correlation. For models with moving average errors and negative serial correlation Z_α and Z_t tests are not recommended. In these cases, the ADF test is preferable. Moreover, the simulation study of Engle and Granger (1987) also suggests using this type of test.

Both Z_α and Z_t tests have no cointegration as the null hypothesis. Park (1990) proposes a test for unit root and cointegration using the variable addition approach by regressing the OLS residuals of the cointegrating regression on the powers of time and testing whether the coefficients involved are jointly zero. This same idea has been used by Bierens and Guo (1993) to test the (trend) stationarity against the unit root hypothesis. However, Park's approach requires consistent estimation of the long-run variance for errors made by the true cointegrating regression by a Newey-West (1987) type estimator, one which reduces a considerable amount of asymptotic power of the test. Furthermore, the tests of Hansen (1992) and Park (1992) are based on a single cointegrating regression, and both tests employ variants of the instrumental variables estimation method of Phillips and Hansen (1990). Finally, Boswijk (1994, 1995) links the single-equation and system approaches by using structural single-equations as a basis for cointegration analysis.

The above approaches test the null or alternative hypothesis of an absence of cointegration, but if the tests indicate the presence of cointegration in systems with three variables or more, we still don't know how many linear independent cointegrating vectors there are. In such cases one may use the approach of Stock and Watson (1988), which is a multivariate extension of the Engle-Granger and Phillips-Ouliaris tests. The basic idea is to linearly transform the k-variate cointegrated process of $\boldsymbol{x}_t$ with say, the h linear independent cointegrating vectors such that the first h components of the transformed $\boldsymbol{x}_t$ are stationary, and the last $k-h$ components, stacked in a vector $\boldsymbol{w}_t$ are integrated. The transformation matrix involved can be estimated using principal components of $\boldsymbol{x}_t$. $\boldsymbol{w}_t$ can then be tested to see whether it is a $k-h$ variate unit root process, using a multivariate version of the ADF test or the Phillips (1987) test. The critical values of this test differ according to whether the initial value $\boldsymbol{x}_0$ is non-zero or not and whether the unit root process $\boldsymbol{x}_t$ has drifted or not.

Other studies on residual based cointegration tests include Park (1990), Bierens and Guo (1993), Hansen (1992) and Park (1992), Phillips and Hansen (1990), Boswijk (1994, 1995), and Stock and Watson (1988).

1.4.2.2 The Johansen Trace and Maximal Eigenvalue Tests

To test whether the variables are cointegrated or not, one of the well-known tests is the Johansen trace test. The Johansen test is used to test for the existence of cointegration and is based on the estimation of the ECM by the maximum likelihood, under various assumptions about the trend or intercepting parameters, and the number h of cointegrating vectors, and then conducting likelihood ratio tests. Assuming that the ECM errors $\boldsymbol{a}_t$ are independent $N_k[\mathbf{0}, \boldsymbol{\Omega}]$ distribution, and given the cointegrating restrictions on the trend or intercept parameters, the maximum likelihood $L_{max}(h)$ is a function of the cointegration rank h. The trace test is based on the log-likelihood ratio $\ln[L_{max}(h)/L_{max}(k)]$, and is conducted sequentially for $h = k$-1,...,1,0. The name comes from the fact that the test statistics involved are the trace (the sum of the diagonal elements) of a diagonal matrix of generalized eigenvalues. This test examines the null hypothesis that the cointegration rank is less than or equal to h, against the alternative that the cointegration rank is greater than h. If the trace is greater than the critical value for a certain rank, then the null hypothesis that the cointegration rank is equal to h is rejected.

Consider a non-stationary cointegrated VAR(p) model

$$(\boldsymbol{I} - \boldsymbol{\phi}_1 \mathrm{B} - \ldots - \boldsymbol{\phi}_p \mathrm{B}^{\mathrm{p}})\boldsymbol{x}_t = \boldsymbol{a}_t$$

where $\boldsymbol{a}_t$ are normally distributed with mean $\mathbf{0}$ and covariance matrix Ω. In a series of influential papers, Johansen (1988, 1991), and Johansen and Juselius (1990) proposed practical full maximum likelihood estimation and testing approaches based on the error correction representation (ECM) in the Equation (1.11).

$$\Delta \boldsymbol{x}_t = \Gamma_0 \boldsymbol{d}_t + \sum_{j=1}^{p-1} \Gamma_j \Delta \boldsymbol{x}_{t-j} + \Pi \boldsymbol{x}_{t-p} + \boldsymbol{a}_t \,. \qquad (1.20)$$

where $\Delta \boldsymbol{x}_t = \boldsymbol{x}_t - \boldsymbol{x}_{t-1}$, $\boldsymbol{d}_t$ is a vector of deterministic variables, such as constant and seasonal dummy variables, $\Gamma_j = -\boldsymbol{I} + \boldsymbol{\phi}_1 + \cdots + \boldsymbol{\phi}_j$, j = 1,⋯, p-1 are $k \times k$, $\Pi = \gamma \boldsymbol{A}'$, $\boldsymbol{A}$ and γ are $k \times h$ parameter matrices, the $\boldsymbol{a}_t$ are i.i.d. $N_k(\mathbf{0}, \Omega)$ errors, and $\det(\boldsymbol{I} - \sum_{j=1}^{p-1} \Gamma_j \mathrm{B}^j)$ has all of its roots outside the unit circle.

The ECM in the Equation (1.20) is based on the Engle-Granger (1987) error correction representation theorem for cointegrated systems, and the asymptotic inference involved is related to the work of Sims, Stock, and Watson (1990). By step-wise concentrating all the parameter matrices in the likelihood function out except for the matrix A, Johansen shows that the maximum likelihood estimator of A can be derived as the solution of a generalized eigenvalue problem. Likelihood ratio tests of hypotheses about the number of cointegrating vectors can then be based on these eigenvalues. Moreover, Johansen (1988) also proposes likelihood ratio tests for linear restrictions on these cointegrating vectors.

Initially, Johansen (1988) considers the case where $\boldsymbol{d_t}$ is absent. Eventually, Johansen (1991) extends his approach to the case where $\boldsymbol{d_t}$ contains an intercept and seasonal dummy variables, and in 1994 he also includes a time trend in $\boldsymbol{d_t}$, but no seasonal dummy variables are allowed. These three cases lead to different null distributions of the likelihood ratio tests of the number of cointegrating vectors. Moreover, possible restrictions on the vector of intercepts or the vector of trend coefficients may also lead to different null distributions. Therefore, the application of Johansen's tests actually requires some prior knowledge about the true parameters of the ECM in the Equation (1.19).

The Johansen test for the existence of cointegration is based on the estimation of the above ECM by the maximum likelihood and is used to test the hypothesis $H_0 : Rank(\Pi) \leq h$, where h is less than k. This formulation shows that I(1) models form nested sequence models $\boldsymbol{H}(0) \subset \cdots \subset \boldsymbol{H}(\mathrm{h}) \subset \cdots \subset \boldsymbol{H}(\mathrm{k})$, where $\boldsymbol{H}(\mathrm{k})$ is the unrestricted VAR model or I(0) model, and $\boldsymbol{H}(0)$ corresponds to the restriction $\Pi = 0$, which is the VAR model for indifferences. Since $\Pi = \gamma A'$, it is equivalent to test that A and γ are of full column rank h, the number of independent cointegrating vectors that forms the matrix $\boldsymbol{A}$. The test has been named the Johansen trace test because the likelihood ratio test statistic is the trace of a diagonal matrix of generalized eigenvalues from Π.

Under some regularity conditions, we can write the cointegrated process $\boldsymbol{x}_t$ as an Error Correction Model (ECM):

$$\Delta \boldsymbol{x}_t = \Gamma_1 \Delta \boldsymbol{x}_{t-1} + \ldots + \Gamma_{p-1} \Delta \boldsymbol{x}_{t-p+1} + \Pi \boldsymbol{x}_{t-p} + \boldsymbol{a}_t \qquad (1.21)$$

where Δ is the difference operator (i.e., $\Delta \boldsymbol{x}_t = \boldsymbol{x}_t - \boldsymbol{x}_{t-1}$), the $\boldsymbol{a}_t$'s are i.i.d. N($\mathbf{0}, \Omega$).

(1.21) can be written as

$$\boldsymbol{Z}_{0t} = \Gamma \boldsymbol{Z}_{1t} + \Pi \boldsymbol{Z}_{pt} + \boldsymbol{a}_t \qquad (1.22)$$

where $\boldsymbol{Z}_{0t} = \Delta \boldsymbol{x}_t$, $\boldsymbol{Z}_{1t} = (\Delta \boldsymbol{x}'_{t-1}, \ldots, \Delta \boldsymbol{x}'_{t-p+1})'$, $\boldsymbol{Z}_{pt} = \boldsymbol{x}_{t-p}$, $\Gamma = (\Gamma_1, \ldots, \Gamma_{p-1})$ and $\boldsymbol{a}_t \sim \text{N}(\mathbf{0}, \Omega)$.

For the fixed value of $\Pi = \gamma \boldsymbol{A}'$, the maximum likelihood estimation consists of a regression of $\boldsymbol{Z}_{0t} - \gamma \boldsymbol{A}' \boldsymbol{Z}_{pt}$ on $\boldsymbol{Z}_{1t}$ giving the normal equations

$$\sum_{t=1}^{n} \boldsymbol{Z}_{0t} \boldsymbol{Z}'_{1t} = \Gamma \sum_{t=1}^{n} \boldsymbol{Z}_{1t} \boldsymbol{Z}'_{1t} + \gamma \boldsymbol{A}' \sum_{t=1}^{n} \boldsymbol{Z}_{pt} \boldsymbol{Z}'_{1t} \qquad (1.23)$$

defining the product moment matrices as

$$\boldsymbol{M}_{ij} = \text{n}^{-1} \sum_{t=1}^{n} \boldsymbol{Z}_{it} \boldsymbol{Z}'_{jt}, \; i, j = 0, 1, p. \qquad (1.24)$$

(1.23) can then be written as

$$\boldsymbol{M}_{01} = \Gamma \boldsymbol{M}_{11} + \gamma \boldsymbol{A}' \boldsymbol{M}_{p1}$$

or

$$\Gamma = \boldsymbol{M}_{01} \boldsymbol{M}_{11}^{-1} - \gamma \boldsymbol{A}' \boldsymbol{M}_{p1} \boldsymbol{M}_{11}^{-1} \qquad (1.25)$$

and the residuals can be obtained by regressing Z_{0t} and Z_{pt} on Z_{1t}.

$$R_{0t} = Z_{0t} - M_{01}M_{11}^{-1}Z_{1t}$$
$$R_{pt} = Z_{pt} - M_{p1}M_{11}^{-1}Z_{1t}.$$

Thus, the likelihood function is proportional to

$$|\Omega|^{n/2} \exp\left\{-\left\{\sum_{t=1}^{n}(R_{0t} - \gamma A'R_{pt})'\Omega^{-1}(R_{0t} - \gamma A'R_{pt})\right\}/2\right\}, \tag{1.26}$$

and this function is minimized for fixed A with

$$\hat{\gamma}(A) = S_{0p}A(A'S_{pp}A)^{-1} \tag{1.27}$$

$$\hat{\Omega}(A) = \left(S_{00} - S_{0p}A(A'S_{pp}A)^{-1}A'S_{p0}\right)/\mathrm{n} \tag{1.28}$$

where $S_{ij} = \mathrm{n}^{-1}\sum_{t=1}^{n} R_{it}R_{jt}' = M_{ij} - M_{i1}M_{11}^{-1}M_{1j}$. Therefore,

$$L_{\max}^{-2/n}(A) = |\hat{\Omega}| = |S_{00} - S_{0p}A(A'S_{pp}A)^{-1}A'S_{p0}| \tag{1.29}$$

$$= |S_{00}|\,|A'S_{pp}A - AS_{0p}S_{00}^{-1}S_{p0}A'| / |A'S_{pp}A|$$

which is based on the general result of

$$\begin{vmatrix} H & B \\ B' & C \end{vmatrix} = |H|\,|C - B'H^{-1}B| = |C|\,|H - BC^{-1}B'|.$$

Johansen (1995, Lemma A.8, p. 224) proves that an equation of

$$|\beta'(M_1 - M_2)\beta| / |\beta' M_1 \beta|$$

can be minimized by solving the following

$$|\lambda M_1 - M_2| = 0$$

where λ is the eigenvalue of M_2 with respect to M_1 with $M_1 = S_{pp}$ and

$M_2 = S_{p0}S_{00}^{1}S_{0p}$. And so, (1.29) is then minimized by the choice $\hat{A} = (\hat{v}_1, \ldots, \hat{v}_h)$ where $\hat{A}$ is the maximum likelihood estimate of cointegrating matrix A, and $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_k)$ are the eigenvectors of the equation

$$|\lambda S_{pp} - S_{p0}S_{00}^{-1}S_{0p}| = 0 \tag{1.30}$$

where λ is the eigenvalue of $S_{p0}S_{00}^{-1}S_{0p}$ with respect to S_{pp} and eigenvectors are normed by $\hat{V}'S_{pp}\hat{V} = I$, and ordered by $\hat{\lambda}_1 > \ldots > \hat{\lambda}_k > 0$. The maximized likelihood function is found from

$$L_{\max}^{-2/n}(\mathrm{h}) = |S_{00}| \prod_{i=1}^{h} (1 - \hat{\lambda}_i). \tag{1.31}$$

Hence, the likelihood ratio statistic for hypothesis $\mathrm{H}_0 : \Pi = \gamma A'$ is given by

$$-2 \ln \Lambda = -n \sum_{i=h+1}^{k} \ln(1 - \hat{\lambda}_i) \tag{1.32}$$

where $\hat{\lambda}_i$ denotes the eigenvalues and are ordered by $\hat{\lambda}_1 > \ldots > \hat{\lambda}_k > 0$. If the test statistics are greater than the critical value for rank *h*, then the null hypothesis that the cointegration rank is equal to *h* is rejected. The statistic $-2 \ln \Lambda$ has the following limiting distribution which can be expressed in terms of a $(k-h)$ – dimensional Brownian motion as

$$\mathrm{tr}\left\{ \int_0^1 (\mathrm{d}Y) Y' \left[\int_0^1 YY' dt \right]^{-1} \int_0^1 Y (\mathrm{d}Y)' \right\}. \tag{1.33}$$

The percentiles of the asymptotic distribution for the trace statistic are tabulated in Johansen (1988, Table 1) using simulation analysis.

An alternative LR statistic, given by

$$-2\ln\Lambda = -n\ln(1-\hat{\lambda}_{h+1}), \qquad (1.34)$$

and called the maximal eigenvalue statistic, examines the null hypothesis of h cointegrating vectors versus the alternative h+1 cointegrating vectors. The asymptotic distribution of this statistic is given by the maximum eigenvalue of the stochastic matrix in

$$\int_0^1 (\mathrm{d}\boldsymbol{Y})\boldsymbol{Y}'\left[\int_0^1 \boldsymbol{Y}\boldsymbol{Y}'\mathrm{d}t\right]^{-1}\int_0^1 \boldsymbol{Y}(\mathrm{d}\boldsymbol{Y})'.$$

Phillips' (1991) efficient error correction modeling approach differs from that of Johansen (1988) in that Phillips specifies the ECM directly on the basis of cointegrating relations $x_{1,t} = \tau x_{2,t} + u_t$ with u_t a stationary zero mean Gaussian process, which leads to an ECM of the form

$$\Delta \boldsymbol{x}_t = \begin{pmatrix} \boldsymbol{I}_h & -\boldsymbol{\tau} \\ 0 & 0 \end{pmatrix} \boldsymbol{x}_{t-p} + \boldsymbol{v}_t. \qquad (1.35)$$

Here h is the number of cointegrating relations and $\boldsymbol{v}_t$ is a stationary Gaussian process with the long run variance matrix $\Omega = \lim_{n\to\infty} Var\left[(1/\sqrt{n})\sum_{t=1}^{n} v_t\right]$. Phillips shows that under the i.i.d. assumption of $\boldsymbol{v}_t$ the maximum likelihood estimator of $\boldsymbol{\tau}$ is efficient, and this efficiency carries over to the case with dependent errors $\boldsymbol{v}_t$ if $\boldsymbol{\tau}$ is estimated by maximum likelihood on the basis of model (1.35) with i.i.d. N($\mathbf{0}$,$\boldsymbol{\Omega}$) errors $\boldsymbol{v}_t$, and provided $\boldsymbol{\Omega}$ is replaced by a consistent estimator. In contrast with Johansen's maximum likelihood method, however, Phillips' efficient maximum likelihood approach has not yet been widely applied in empirical research, possibily due to the fact that the limited distribution of the maximum likelihood estimator of the matrix τ depends on the long run variance matrix Ω.

1.4.3 Cointegration and Temporal Aggregation

When the frequency of data generation is lower than that of data collection, temporal aggregation arises so that only some function of realizations is observable. Most of the literature is focused on the effects of temporal aggregation in the univariate time series; however, many properties of interest such as exogeneity, causality, and cointegration can only be defined in a multivariate context. Marcellino (1996) proves that both the number and composition of cointegration vectors are the same after aggregation for stock variables. Marcellino (1999) also theoretically shows that time aggregation may increase the local power of cointegration tests when the aggregate variables are obtained by systematic sampling. Haug (2002) studied the effects of time aggregation and the role of spanning data on the power of commonly used univariate and multivariate cointegration tests by using the Monte Carlo method. His results show that size distortions, caused by temporal aggregation, significantly affect relative test performance. These studies either use systematic sampling or do not take into consideration the fact that the form of the process changes after temporal aggregation when flow variables are used.

The Johansen trace tests seem to be the best choice for testing cointegration for a multivariate time series. However, for an aggregate series, the test statistics and its limiting distribution are unknown. Therefore, through this research we find a proper cointegration test statistic and its limiting distribution for an aggregate time series.

1.5 Causality and Temporal Aggregation

In the applied time series analysis, data are usually sums or averages that become available less frequently than the generating process, and they are routinely used to test causality between variables. Tiao and Wei (1976) are the first researchers to show the result that the non-causal relationship turns out to be a causal one after temporal aggregation for flow variables. Wei (1982) reveals the effects of temporal aggregation on parameter estimation in a finite distribution lag model through the least square procedure; he points out that the loss in efficiency due to aggregation could not be disregarded. This loss depends on both the level of aggregation and the nature of the input variable. If the input variable is negatively correlated, the loss is more severe. A Monte Carlo simulation is conducted by Cunningham and Vilasuso (1995) to investigate the distortion effects of temporal aggregation; as the span of aggregation widens, the chance of detecting the true causality decreases. Also, the bivariate time series x_{1t} and x_{2t} is considered by them to be a stable autoregressive process

$$\begin{bmatrix} \phi_{11}(B) & \phi_{12}(B) \\ \phi_{21}(B) & \phi_{22}(B) \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

where $(a_{1t}, a_{2t}) \sim N(0, \Omega)$, Ω is a diagonal matrix. Cunningham and Vilasuso formed temporal aggregation by averaging basic observations on non-overlapping intervals and obtaining

$$X_{1T} = \frac{1}{m} \sum_{j=0}^{m-1} x_{1,Tm-j} \text{ and } X_{2T} = \frac{1}{m} \sum_{j=0}^{m-1} x_{2,Tm-j} .$$

While systematic sampling preserves the direction of causality, they point out that temporal aggregation could change the true one-sided causal relationship into a two-sided feedback system based on the work of Wei (1982). According to their simulation results, the probability of failing to reject false hypotheses approaches 90% and at short aggregation intervals, temporal aggregates are between two and ten times more likely to fail to detect a true causal relationship. In 1996, Mamingi presents the existence of Granger causality distortion on error correction models (ECMs) due to aggregation over time, again using the Monte Carlo simulation. He finds that distortion depends on the degree of cointegration, the data span, the sample size, and the type of aggregation. He also mentions that causality distortion is a change of x_t causality in basic ECMs into another type of causality in aggregated ECMs. According to his simulation results, systematic sampling creates less Granger causality distortion than temporal aggregation, and a large data span is more harmful to true Granger causality between variables than a large sample size is. However, there are no theoretical results on aggregation effects on ECM models. Marcellino (1999) derives the generating mechanism of a temporally aggregated process when the basic series follows a vector ARIMA process and studies the effect of temporal aggregation on a set of characteristics, such as causality. It was discovered that Granger non-causality is usually lost after temporal aggregation.

The earlier studies of aggregation on multiple time series were mostly based upon simple distributed lag models. Recent studies have started to use vector processes, but most of these studies support their findings through Monte Carlo experiments. In this study, we examine the use of aggregate time series in vector time series modeling. First, we derive the theoretical and numerical vector autoregressive moving average model for aggregate variables. Next, we examine various tests and investigate the effects of temporal aggregation on the causal relationship of these variables. The causality found between aggregate variables may not be the real phenomenon of the underlying model. The effect of temporal aggregation on the non-causality test in cointegrated systems is then investigated.

In a vector autoregressive process, the Granger non-causality of one set of variables for another is characterized by having no constraints on the autoregressive coefficients. If the process is stationary, the test for non-causality is usually performed using Wald (or likelihood ratio) tests which are asymptotically chi-squared. Phillips and Durlauf (1986), Park and Phillips (1988, 1989), Sims, Stock, and Watson (1990), Lütkepohl and Reimers (1992), Toda and Phillips (1993), and Caporale and Pittis (1999) have all shown that the asymptotic theory of Wald tests is much more complex in cointegrated systems. Sims, Stock and Watson (1990) worked on trivariate systems in the VAR process and showed that the Wald test has limiting chi-squared distribution if the time series are cointegrated. However, Toda and Phillips (1993) showed that without explicit information on the number of unit roots in the system and the rank of certain sub-matrices in the cointegration space, it is impossible to determine the appropriate limit theory; even when such information is available the limit theory involves nuisance parameters and non-standard distributions. Mosconi and Giannini (1992) suggest a likelihood ratio test which is more efficient by imposing the cointegration constraints under both the null and the alternative hypotheses. Therefore we will use their approach to test causality in cointegrated systems for aggregates.

A time series $\{x_{1t}\}$ is said to cause another time series $\{x_{2t}\}$ in the sense defined by C. W. J. Granger (1969) if the present value of x_2 can be better predicted by using the past values of x_1 and x_2 rather than using only the past values of x_2.

Let the k dimensional vector autoregressive moving average process x_t be partitioned into two vectors $x_t = (x'_{1,t}, x'_{2,t})'$ where $x_{1,t} = (y_{1t},...,y_{k_1t})'$ and $x_{2,t} = (y_{k_1+1,t},...,y_{k_1+k_2,t})'$ are k_1 and k_2 dimensional vectors respectively, and also $k = k_1 + k_2$. A time series $\{x_{1,t}\}$ is said to cause another time series $\{x_{2,t}\}$ if the present value of x_2 can be better predicted by using the past values of x_1 and x_2 rather than by using only the past values of x_2. Assume the following axioms: the cause cannot come after the effect, the cause contains some unique information which affects the future value, and while the strength of casual relations varies over time, their existence is time invariant (Granger 1980, 1998).

Let I_t be information set from the vector series $x_{1,t}$ and $x_{2,t}$ up to time t, i.e., $I_t = \{x_{1,s}, x_{2,s} : s \le t\}$. For any information set I_t, the best mean square linear predictor of $y_{i,t}$ is denoted by $P(y_{i,t} | I_t)$. The predictor $P(y_{i,t} | I_t)$, is the orthogonal projection $y_{i,t}$ on the Hilbert space spanned by the variables in I_t. For Gaussian processes, we use $P(y_{i,t} | I_t) = E(y_{i,t} | I_t)$. The best predictor of $x_{2,t}$ is the vector $P(x_{2,t} | I_t) = \left[P(y_{k_1+1,} | I_t),...,P(y_{k_1+k_2,t} | I_t)\right]'$ that corresponds with the vector of prediction errors given by $\varepsilon_{2,t}(x_{2,t} | I_t) = \left[\varepsilon_{k_1+1,t}(y_{k_1+1,t} | I_t),...,\varepsilon_{k_1+k_2,t}(y_{k_1+k_2,t} | I_t)\right]'$ where $\varepsilon_{i,t}(y_{i,t} | I_t) = y_{i,t} - P(y_{i,t} | I_t)$, and the covariance matrix of $\varepsilon_{2,t}$ is $\Omega(x_{2,t} | I_t)$. Hence, the definition of non-causality is the vector $\boldsymbol{x}_{1,t}$ does not cause $\boldsymbol{x}_{2,t}$ if

$$\Omega(\boldsymbol{x}_{2,t} \mid I_t) = \Omega(\boldsymbol{x}_{2,t} \mid I_t \setminus \{\boldsymbol{x}_{1,s} \mid s \le t\}) \tag{1.36}$$

where $I_t \setminus \{x_{1,s} \mid s \le t\}$ denotes the information set available at time t containing only the vector series $\boldsymbol{x}_{2,t}$.

Most of the literature on causality tests are based on bivariate models or vector autoregressive processes. Newbold (1982) uses a likelihood ratio to test non-causality in a bivariate ARMA model. Eberts and Steece (1984), Newbold and Hotoop (1986) again consider the Wald, likelihood ratio, and score tests for bivariate models. Tjostheim (1981) and Hsiao (1982) give a general formulation and non-causality conditions for trivariate models. Boudjellaba, Dufour, and Roy (1992) derive the necessary and sufficient conditions for non-causality between two vectors of variables and introduce Granger causality tests. They develop a causality analysis for general vector processes.

Consider the stationary and invertible k-dimensional VARMA (p, q) process

$$\Pi_p(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{\varphi}_q(\mathrm{B})\boldsymbol{a}_t, \tag{1.37}$$

where $\Pi(\mathrm{B}) = \boldsymbol{I} - \Pi_1\mathrm{B} - \ldots - \Pi_p\mathrm{B}^{\mathrm{p}}$ and $\varphi(\mathrm{B}) = \boldsymbol{I} - \varphi_1\mathrm{B} - \ldots - \varphi_q\mathrm{B}^{\mathrm{q}}$. The $\boldsymbol{a}_t$'s are uncorrelated random vectors with mean $\mathbf{0}$ and the non-singular covariance matrix $\boldsymbol{\Omega}$. Assume that the parameters in $\Pi(\mathrm{B})$ and $\varphi(\mathrm{B})$ are uniquely defined and the process $\boldsymbol{x}_t$ is partitioned into two vectors $\boldsymbol{x}_t = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t})'$, where $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ are k_1 and k_2 dimensional vectors respectively with $k_1 + k_2 = k$. Then, $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if

$$\det(\Pi_1(z), \varphi_{(2)}(z)) = 0, \tag{1.38}$$

where $\Pi_1(z)$ is the first k_1 column of $\Pi(z)$ and $\varphi_{(2)}(z)$ is the matrix of $\varphi(z)$ without its last k_2 columns (Boudjellaba, Dufour and Roy, 1992).

More explicitly, we can write (1.37) as

$$\begin{pmatrix} \Pi_{11}(B) & \Pi_{12}(B) \\ \Pi_{21}(B) & \Pi_{22}(B) \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} \varphi_{11}(B) & \varphi_{12}(B) \\ \varphi_{21}(B) & \varphi_{22}(B) \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}. \tag{1.39}$$

Boudjellaba, Dufour and Roy (1992) prove that if the stationary VARMA process is invertible with det($\varphi_{11}(z)$)$\neq 0$ for all $z \in C$ such that $|z| \leq 1$, then (1.38) states that x_1 does not cause x_2 if and only if

$$\Pi_{21}(z) - \varphi_{21}(z)\varphi_{11}(z)^{-1}\Pi_{11}(z) = 0. \tag{1.40}$$

Similarly, Lütkepohl (1991) investigates the necessary and sufficient rules for non-causality between two groups of stationary time series variables. Consider the non-stationary aggregate series x_t with the MA representation

$$x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \Psi_{11}(B) & \Psi_{12}(B) \\ \Psi_{21}(B) & \Psi_{22}(B) \end{bmatrix} a_t \tag{1.41}$$

where $x_{i,t}$; $i=1,2$ are $k_i \times 1$, $i=1,2$ vector, a_t is a k-dimensional normal white noise vector with mean $\mathbf{0}$ and a covariance matrix Ω, and $\Psi_{ij}(B) = \sum_{\ell=0}^{\infty} \Psi_{ij,\ell} B^{\ell}$; $i,j = 1,2$. Then, x_2 does not cause x_1 iff $\Psi_{12}(B) = \mathbf{0}$. Similarly, x_1 does not cause x_2 iff $\Psi_{21}(B) = \mathbf{0}$.

1.5.1 Testing Causality

In 1969 and 1980, Granger defines causality in terms of predictability and discusses methods to test it. Later, Sims (1972) proves that non-causality from x_1 to x_2 is equivalent to the hypothesis that the regression coefficients of future x_2 are zero in the regression of x_1 in future, present and past x_2. Following Sims' method, many writers offered different causality tests; for example, Mehra (1978) and Sims (1975). However, the most commonly used Granger causality tests is the Wald test and the Likelihood Ratio (LR) test. In 1982, Geweke developed a measure of linear causality between two variables by using the linear projection of variables. He also solved the inference problem of these measures by using non-central chi-square distribution.

1.5.1.1 Wald and Likelihood Ratio Tests for Testing Non-Causality

Given a series of n observations $x = (x_1', x_2')'$, we will consider the following hypothesis on causality:

$$H_0: x_1 \text{ does not cause } x_2$$

$$H_A: x_1 \text{ causes } x_2.$$

To perform a Likelihood Ratio test, we first construct a multivariate VARMA model by letting δ be the vector of all AR and MA parameters and δ_1 be the $w \times 1$ vector of constraints on δ. Denote the restrictions as

$$R_j(\delta_1) = 0, \text{ j=1,2,...,K}$$

where $K \le w$. We will then use the following test statistics:

Wald statistic:

$$\xi_W = nR(\hat{\delta}_1)'\left[T(\hat{\delta}_1)'V(\hat{\delta}_1)T(\hat{\delta}_1)\right]^{-1} R(\hat{\delta}_1) \sim \chi^2_K \text{ asymptotically}, \qquad (1.42)$$

where $\hat{\delta}_1$ is the maximum likelihood estimate of δ, $R(\delta_1) = \left(R_1(\delta_1), ..., R_K(\delta_1)\right)'$, $T(\hat{\delta}_1)$ is a matrix of derivatives of $R(\delta_1)$ at $\hat{\delta}_1$, and $V(\hat{\delta}_1)$ is the asymptotic covariance matrix of $\sqrt{n}(\hat{\delta}_1 - \delta_1)$. The Wald test is easy to apply because it only uses the maximum likelihood estimators of the constrained parameters of the full model and does not require the estimation of unconstrained parameters.

Likelihood Ratio statistic:

$$\xi_{\mathrm{LR}} = 2(\mathrm{L}(\hat{\delta}, x) - \mathrm{L}(\hat{\delta}^*, x)) \sim \chi^2_{\mathrm{K}} \text{ asymptotically,} \tag{1.43}$$

where $\mathrm{L}(\delta, x)$ is the logarithm of the likelihood, $\hat{\delta}^*$ is the MLE of δ under constraints $R_j(\delta_1)$, and $\hat{\delta}$ is the unconstraint MLE.

Both ξ_W and ξ_{LR} are asymptotically equivalent and follow χ^2_K distribution where K is the number of restrictions under the null hypothesis of non-causality (Basawa, Billard, Srinivasan, 1984).

1.5.1.2. Testing Causality in Cointegrated Systems

Testing for Granger non-causality in cointegrated time series has been the subject of considerable recent research. The first result that naturally emerges from this subject is the existence of 'long-run' causality in at least one direction (Granger, 1988), whereas cointegration is represented by a bivariate error–correction model. The extension of this result to more than two variables is fairly straightforward under the existence of one cointegrating relation. In fact, the two-step procedure introduced by Engle and Granger (1987) was all that was needed to test non-causality hypotheses. In empirical literature the Wald test computes from an unrestricted vector autoregressive (VAR) model that appears frequently. Mosconi and Giannini (1992) suggest a likelihood ratio test that uses ECM.

Given a non-stationary, k-dimensional, cointegrated $I(1)$ series $\boldsymbol{x}_t$ that follows a vector autoregressive VAR(p) process:

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t, \tag{1.44}$$

where $\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\phi}_1\mathrm{B} - \ldots - \boldsymbol{\phi}_p\mathrm{B}^{\mathrm{p}}$ and $\boldsymbol{a}_t$'s are i.i.d. k-dimensional normal random vectors with mean $\mathbf{0}$ and the variance-covariance matrix $\boldsymbol{\Omega}$ such that $A'\boldsymbol{x}_t$ is $I(0)$ and $rank(A)$ = h. The error correction model of (1.44) can be written as

$$\Delta\boldsymbol{x}_t = \sum_{i=1}^{p-1}\Gamma_i\Delta\boldsymbol{x}_{t-i} + \Pi\boldsymbol{x}_{t-p} + \boldsymbol{E}_T \tag{1.45}$$

where $\Gamma_i = -\boldsymbol{I} + \boldsymbol{\phi}_1 + \cdots + \boldsymbol{\phi}_i$, $i = 1,\ldots,p-1$, $\Pi = -\boldsymbol{\phi}_p(1) = \gamma A'$ for some γ.

Let's partition $\boldsymbol{x}_t$ as $\boldsymbol{x}_t = (\underset{1\times k_1}{\boldsymbol{x}'_{t,1}}, \underset{1\times k_2}{\boldsymbol{x}'_{t,2}})'$ where $\mathrm{k=k_1+k_2}$. Equations (1.44) and (1.45) are given by

$$\begin{bmatrix} \boldsymbol{\phi}_{p,11}(\mathrm{B}) & \boldsymbol{\phi}_{p,12}(\mathrm{B}) \\ \boldsymbol{\phi}_{p,21}(\mathrm{B}) & \boldsymbol{\phi}_{p,22}(\mathrm{B}) \end{bmatrix}\boldsymbol{x}_t = \boldsymbol{a}_t$$

and

$$\Delta\boldsymbol{x}_t = \sum_{i=1}^{p-1}\begin{bmatrix} \Gamma_{i,11} & \Gamma_{i,12} \\ \Gamma_{i,21} & \Gamma_{i,22} \end{bmatrix}\Delta\boldsymbol{x}_{t-i} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}\boldsymbol{x}_{t-p} + \boldsymbol{E}_T.$$

The MA representation is then given by

$$(1-\mathrm{B})\boldsymbol{x}_t = \Psi(\mathrm{B})\boldsymbol{a}_t$$

where $\Psi(\mathrm{B}) = \boldsymbol{I} + \Psi_1\mathrm{B} + \Psi_2\mathrm{B}^2 + \cdots = \begin{bmatrix} \Psi_{11}(\mathrm{B}) & \Psi_{12}(\mathrm{B}) \\ \Psi_{21}(\mathrm{B}) & \Psi_{22}(\mathrm{B}) \end{bmatrix}$. So, $\boldsymbol{x}_{t,2}$ does not cause $\boldsymbol{x}_{t,1}$ if $\Psi_{12}(\mathrm{B}) = \boldsymbol{0}$. This means that $\boldsymbol{\phi}_{p,12}(\mathrm{B}) = \boldsymbol{0}$ which implies $\Gamma_{i,12} = \boldsymbol{0}$ and $\Pi_{12} = \boldsymbol{0}$. In this framework, $x_{t,2}$ does not cause $x_{t,1}$ if the hypothesis

$$\text{H: } U'\Gamma V = 0, \quad U'\Pi U_{\perp} = 0$$

holds where $U = \left[0'_{k-k_2,k_2} \quad I'_{k_2} \right]'_{k \times k_2}$, $\Gamma = \left[\Gamma'_1 \quad \cdots \quad \Gamma'_{p-1} \right]'$, $V = I_{p-1} \otimes U_{\perp}$, and $U_{\perp} = \left[I'_{k_1} \quad 0'_{k-k_1,k_1} \right]'_{k \times k_1}$.

Mosconi and Giannini (1992) prove in their Theorem 1 that given any reduced rank matrix, $\Pi = \gamma A'$, $U'\Pi U_{\perp}$ is equal to zero matrix iff

$$\gamma = \left[U_{\perp}\gamma_{11} \quad | \gamma_2 \right] \text{ and } A = \left[A_1 \quad | UA_{22} \right] \tag{1.46}$$

where γ_{11} is $k_1 \times h_1$, γ_2 is $k \times h_2$, A_1 is $k \times h_1$, and A_{22} is $k_2 \times h_2$ with $h = h_1 + h_2$.

Let's partition $\gamma_2 = \left[\gamma'_{12} \quad \gamma'_{22} \right]'$ and $A_1 = \left[A'_{11} \quad A'_{21} \right]'$ where γ_{12} is $k_1 \times h_2$, γ_{22} is $k_2 \times h_2$, A_{11} is $k_1 \times h_1$, and A_{21} is $k_2 \times h_1$. This means that

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Then,

$$\Pi = \gamma A' = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \begin{bmatrix} A'_{11} & A'_{21} \\ 0 & A'_{22} \end{bmatrix} = \begin{bmatrix} \gamma_{11}A'_{11} & \gamma_{11}A'_{21} + \gamma_{12}A'_{22} \\ 0 & \gamma_{22}A'_{22} \end{bmatrix}$$

which means that h_1 is rank (Π_{11}) and h_2 is rank (Π_{22}).

The null hypothesis of non-causality in aggregate cointegrated systems, $H_0(h, h_1, h_2)$ defined by Mosconi and Giannini (1992) then becomes

$$H_0(h, h_1, h_2)\text{: } U'\Gamma V = 0, \quad U'\Pi U_{\perp} = 0, \quad \Pi = \gamma A',$$

where γ is the full rank $k \times h$ adjustment matrix and A is the full rank $k \times h$

cointegrating matrix and the alternative hypothesis is $H_A(h)$: $\Pi = \gamma A'$. By applying the cointegration restrictions under both the null and the alternative hypotheses, the more efficient test can be obtained.

Under the null hypothesis $H_0(h,h_1,h_2)$, the equation (1.45) can be rewritten as

$$\Delta x_t = \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + (U_\perp \gamma_{11} A_1' + \gamma_2 A_{22}' U') x_{t-p} + e_t . \qquad (1.47)$$

If we let $z_{0t} = \Delta x_t$, $z_{1t} = \left(\Delta x_{t-1}', \dots, \Delta x_{t-p+1}'\right)'$, $z_{pt} = x_{t-p}$ and $\Gamma = \left(\Gamma_1, \dots, \Gamma_{p-1}\right)$. The equation (1.47) then becomes

$$z_{0t} = \Gamma z_{1t} + (U_\perp \gamma_{11} A_1' + \gamma_2 A_{22}' U') z_{pt} + e_t , \qquad (1.48)$$

where $U'\Gamma V = 0$.

Mosconi and Giannini (1992) suggest an iterative algorithm to find the estimates and the maximized likelihood under the null hypothesis of non-causality. If we define $\hat{\gamma}_{11,i}, \hat{A}_{1,i}, \hat{\gamma}_{2,i}, \hat{A}_{22,i}, \hat{\Gamma}_i$ and $\hat{\Omega}_i$, the ith step estimates the corresponding parameters. Given $\hat{\gamma}_{11,i}, \hat{A}_{1,i}$ and $\hat{\Gamma}_i$, the ith step estimates for $\hat{\gamma}_{2,i}, \hat{A}_{22,i}$ and $\hat{\Omega}_i$ are obtained. Finally, given $\hat{\gamma}_{2,i}$ and $\hat{A}_{22,i}$, the (i+1)th step estimates of γ_{11}, A_1 and Γ are derived. The convergence criterion can be expressed in increments of the log-likelihood. If the maximized likelihood after convergence is denoted by $\max_{H_0(h,h_1,h_2)} L\left[\Gamma, \Pi; x_1, \cdots, x_n\right]$, then the likelihood ratio test is given by

$$-2\ln \frac{\max_{H_0(h,h_1,h_2)} L\left[\Gamma, \Pi; x_1, \cdots, x_n\right]}{\max_{H_A(h)} L\left[\Gamma, \Pi; x_1, \cdots, x_n\right]} \qquad (1.49)$$

where $\max_{H_A(h)} L[\Gamma,\Pi;x_1,\cdots,x_n]$ is the maximized likelihood function in equation (1.31) obtained by Johansen (1988). Based on Johansen's results, Mosconi and Giannini (1992) indicate that it is asymptotically distributed as Chi-squared, and they compute the degrees of freedom to be $kh-k_1h_1-k_2h_2-h_1h_2$ when $p=1$ and $kh-k_1h_1-k_2h_2-h_1h_2+k_1k_2(p-1)$ when $p>1$.

1.6 Representation of Multiplicative Vector Autoregressive Moving Average Processes

Many business and economic time series have a seasonal behavior. By seasonal behavior it is meant that the recurrence of some recognizable pattern after some regular interval is called the seasonal period and denoted by s. Some applied researchers prefer to use officially adjusted time series; however, many studies emphasize the importance of using unadjusted time series. Sims (1974), Wallis (1974), and Ghysels (1988) especially claimed that the official seasonal adjustment results in biases in the estimated dynamic relationships and provides a weak relationship between seasonally adjusted series of production, sales, and inventories. Moreover, Raynauld and Simonato (1993) claim that there is a loss of degrees of freedom coming from the seasonal adjustment process when a model with an officially adjusted series is estimated. Hence, a model for unadjusted seasonal series is called for, and the best way to present this seasonal behavior mathematically is by using multiplicative models. Although a non-multiplicative representation can also be used, it is less efficient because it normally involves an estimation of a larger number of parameters.

For a univariate seasonal time series x_t, the multiplicative autoregressive and moving average model have been found to be very useful in describing the seasonal time series

$$\phi_p(\mathrm{B})\Phi_P(\mathrm{B}^s)x_t = \theta_0 + \theta_q(\mathrm{B})\Theta_Q(\mathrm{B}^s)a_t, \tag{1.50}$$

where B is the back shift operator, $\mathrm{B}x_t = x_{t-1}$,

$$\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \cdots - \Phi_P B^{Ps},$$

$$\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q,$$

$$\Theta_Q(B^s) = 1 - \Theta_1 B^s - \cdots - \Theta_Q B^{Qs},$$

and a_t is the Gaussian white noise process with mean 0 and constant variance σ_a^2. The model is often denoted as $ARMA(p,q)\times(P,Q)_s$, whereas s is the seasonal period. For our study, we will denote the model as $ARMA(p)(P)_s(q)(Q)_s$. When the $\boldsymbol{x}_t$ is an k-dimensional vector, the natural extension is the following multiplicative vector autoregressive and moving average model

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{\Phi}_P(\mathrm{B}^s)\boldsymbol{x}_t = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_q(\mathrm{B})\boldsymbol{\Theta}_Q(\mathrm{B}^s)\boldsymbol{a}_t, \tag{1.51}$$

where

$$\boldsymbol{\phi}_p(B) = \mathbf{I} - \boldsymbol{\phi}_1 B - \cdots - \boldsymbol{\phi}_p B^p,$$

$$\boldsymbol{\Phi}_P(B^s) = \mathbf{I} - \boldsymbol{\Phi}_1 B^s - \cdots - \boldsymbol{\Phi}_P B^{Ps},$$

$$\boldsymbol{\theta}_q(B) = \mathbf{I} - \boldsymbol{\theta}_1 B - \cdots - \boldsymbol{\theta}_q B^q,$$

and

$$\Theta_Q(B^s) = \mathrm{I} - \Theta_1 B^s - \cdots - \Theta_Q B^{Qs},$$

$\mathbf{I}$ is the $k \times k$ identity matrix, $\boldsymbol{\phi}_i$, $\boldsymbol{\Phi}_j$, $\boldsymbol{\theta}_k$, $\boldsymbol{\Theta}_\ell$ are $k \times k$ coefficient matrices, and $\boldsymbol{a}_t$ is the vector Gaussian white noise process with mean vector $\mathbf{0}$ and the constant variance-covariance matrix $\boldsymbol{\Omega}$. This vector model will be denoted as $VARMA(p)(P)_s(q)(Q)_s$. When p = 0 and P = 0, the model is referred to as a multiplicative vector moving average model of order q and Q with a seasonal period s, and it is shortened to $VMA(q)(Q)_s$. When q = 0 and Q = 0, the model is referred to as a multiplicative vector autoregressive model of order p and P with a seasonal period s, and is shortened to $VAR(p)(P)_s$. We also assume that the zeroes of the determinantal polynomials $|\boldsymbol{\phi}_p(B)\boldsymbol{\Phi}_P(B^s)|$ and $|\boldsymbol{\theta}_q(B)\boldsymbol{\Theta}_Q(B^s)|$ are all outside the unit circle.

Because of the commutative nature of scalars, for a univariate time series, model (1.50) can also be written as

$$\Phi_P(\mathrm{B}^s)\phi_p(\mathrm{B})x_t = \theta_0 + \Theta_Q(\mathrm{B}^s)\theta_q(\mathrm{B})a_t. \tag{1.52}$$

Can this operation be extended to the vector process given in model (1.51)? This problem arises in the literature and yet, surprisingly, no one has ever paid any attention to this issue. For example, in studying the likelihood function for a multiplicative VARMA model, Hillmer and Tiao (1979) write a multiplicative vector moving average $VMA(q)(Q)_s$ model as

$$\boldsymbol{x}_t = \boldsymbol{\theta}_q(\mathrm{B})\boldsymbol{\Theta}_Q(\mathrm{B}^s)\boldsymbol{a}_t. \tag{1.53}$$

On the other hand, in studying the algorithm of the exact likelihood for a vector process, Ansley (1979) uses the following representation

$$x_t = \Theta_Q(B^s)\theta_q(B)a_t, \tag{1.54}$$

which can be denoted as $VMA(Q)_s(q)$. Is $VMA(q)(Q)_s$ the same as $VMA(Q)_s(q)$? More precisely, can both models (1.53) and (1.54) be used to describe the same vector time series? In the literature, studies on representation of multiplicative VARMA processes are nonexistent. Therefore, this research aims to fill such a gap.

CHAPTER 2

TEMPORAL AGGREGATION OF MULTIVARITE AUTOREGRESSIVE MOVING AVERAGE PROCESSES

2.1 Introduction

The time series data used are typically sums or averages of data that is frequently more generated than the reporting interval. Obviously, averaging smoothes the data, but this changes the time series properties at all frequencies. Most of the theoretical literature has focused on the univariate autoregressive (AR) processes or autoregressive integrated moving average (ARIMA) processes for example, Brewer (1973), Wei (1981), and Weiss (1984). Temporal aggregation of multivariate processes is clearly more interesting because many properties of interest such as causality and cointegration can only be studied through multivariate processes.

In this chapter we will analyze some properties of vector time series under temporal aggregation and derive some vector ARIMA models for temporal aggregates, which we will need in the later chapters.

2.2 Derivation of Some Vector Autoregressive Moving Average Processes for Temporal Aggregates

Proposition 2.1 Let $\boldsymbol{x}_t$ be a zero mean basic vector time series following a VMA(1) process:

$$\boldsymbol{x}_t = \boldsymbol{a}_t - \boldsymbol{\theta} \boldsymbol{a}_{t-1}, \qquad (2.1)$$

where $\boldsymbol{a}_t$ vector is a sequence of i.i.d. random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}$. Then the aggregate time series defined by $\boldsymbol{X}_T = (1+B+...+B^{m-1})\boldsymbol{x}_{mT}$ will follow a VMA(1) process

$$\boldsymbol{X}_T = \boldsymbol{E}_T - \boldsymbol{\Theta}\boldsymbol{E}_{T-1} \tag{2.2}$$

where $\boldsymbol{E}_T$ is a sequence of i.i.d. random variables with mean vector $\mathbf{0}$ zero, the covariance matrix $\boldsymbol{\Omega}_E$, the moving average parameter $\boldsymbol{\Theta}$, and the covariance matrix $\boldsymbol{\Omega}_E$ are determined as follows:

(i) If $m = 1$, then

$$\boldsymbol{\Theta} = \boldsymbol{\theta} \text{ and } \boldsymbol{\Omega}_E = \boldsymbol{\Omega}.$$

(ii) If $m > 1$, then $\boldsymbol{\Theta}$ will be the solution of the following quadratic matrix equation

$$\boldsymbol{\Theta}^2\boldsymbol{\Gamma}_1 + \boldsymbol{\Theta}\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1' = \mathbf{0},$$

where $\Gamma_0 = \text{Var}(\boldsymbol{X}_T)$ and $\Gamma_1 = \text{Cov}(\boldsymbol{X}_T, \boldsymbol{X}_{T+1})$, and

$$\boldsymbol{\Omega}_E = -\boldsymbol{\Gamma}_1(\boldsymbol{\Theta}')^{-1}.$$

Proof:

(i) If $m = 1$, then $\boldsymbol{X}_T = \boldsymbol{x}_t$. Consequently,

$$\boldsymbol{\Theta} = \boldsymbol{\theta} \text{ and } \boldsymbol{\Omega}_E = \boldsymbol{\Omega}. \tag{2.3}$$

(ii) If $m > 1$, by multiplying $(1 + B + ... + B^{m-1})$ on both sides of (2.1) we obtain

$$(1+B+...+B^{m-1})\boldsymbol{x}_t = (1+B+...+B^{m-1})(\boldsymbol{a}_t - \boldsymbol{\theta}\boldsymbol{a}_{t-1}) = (1+B+...+B^{m-1})(\boldsymbol{I} - \boldsymbol{\theta}B)\boldsymbol{a}_t.$$

By changing t to mT

$$\boldsymbol{X}_T = (1+B+...+B^{m-1})(\boldsymbol{I} - \boldsymbol{\theta}B)\boldsymbol{a}_{mT},$$

we obtain

$$\Gamma_0 = \mathrm{Var}(\boldsymbol{X}_T) = \mathrm{Var}((1+\mathrm{B}+...+\mathrm{B}^{m-1})(\boldsymbol{I}-\theta\mathrm{B})\boldsymbol{a}_{mT})$$
$$= \mathrm{Var}((\boldsymbol{I}+(\boldsymbol{I}-\theta)\mathrm{B}+...+(\boldsymbol{I}-\theta)\mathrm{B}^{m-1}-\theta\mathrm{B}^{m})\boldsymbol{a}_{mT}).$$

Hence,

$$\Gamma_0 = \Omega + (m-1)(\mathrm{I}-\theta)\Omega(\mathrm{I}-\theta)' + \theta\Omega\theta'. \tag{2.4}$$

Next,

$$\Gamma_1 = \mathrm{Cov}(\boldsymbol{X}_T, \boldsymbol{X}_{T+1}) = \mathrm{E}(\boldsymbol{X}_T \boldsymbol{X}'_{T+1})$$
$$= \mathrm{E}\left\{\left((1+\mathrm{B}+...+\mathrm{B}^{m-1})(\boldsymbol{I}-\theta\mathrm{B})\boldsymbol{a}_{mT}\right)\left((1+\mathrm{B}+...+\mathrm{B}^{m-1})(\boldsymbol{I}-\theta\mathrm{B})\boldsymbol{a}_{mT}\right)'\right\}.$$

Hence,

$$\Gamma_1 = -\Omega\theta'. \tag{2.5}$$

Also,

$$\Gamma_s = \mathrm{Cov}(\boldsymbol{X}_T, \boldsymbol{X}_{T+s}) = \boldsymbol{0}, s > 1.$$

Thus, $\{\boldsymbol{X}_T\}$ is a vector MA(1) process. For the VMA(1) process it is known that

$$\Gamma_0 = \Omega_E + \Theta\Omega_E\Theta', \tag{2.6}$$

and

$$\Gamma_1 = -\Omega_E\Theta'. \tag{2.7}$$

By using this information, the parameter Θ and covariance matrix Ω_E can be obtained as a quadratic matrix equation on Θ

$$\Theta^2\Gamma_1 + \Theta\Gamma_0 + \Gamma_1' = \boldsymbol{0}, \tag{2.8}$$

where Γ_0 and Γ_1 are given in (2.4) and (2.5), respectively. After obtaining the parameter matrix Θ, we can use (2.7) to find Ω_E as

$$\Omega_E = -\Gamma_1(\Theta')^{-1}. \tag{2.9}$$

These results show that if the basic time series follows the VMA(1) process, then the aggregate time series also follows this process, but with different parameters.

Proposition 2.2 Let $\boldsymbol{x}_t$ be a zero mean basic time series following a stationary VAR(1) process:

$$\boldsymbol{x}_t - \boldsymbol{\phi}\boldsymbol{x}_{t-1} = \boldsymbol{a}_t\ , \tag{2.10}$$

where the $\boldsymbol{a}_t$ vector is a sequence of i.i.d. random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}$. Then the aggregate time series defined by $\boldsymbol{X}_T = (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})\boldsymbol{x}_{mT}$ will follow a VARMA (1,1) process

$$\boldsymbol{X}_T - \boldsymbol{\Phi}\boldsymbol{X}_{T-1} = \boldsymbol{E}_T - \boldsymbol{\Theta}\boldsymbol{E}_{T-1}\ , \tag{2.11}$$

where the $\boldsymbol{E}_T$ vector is a sequence of i.i.d. random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}_E$. The autoregressive parameter $\boldsymbol{\Phi}$, the moving average parameter $\boldsymbol{\Theta}$, and the covariance matrix $\boldsymbol{\Omega}_E$ are determined as follows:

(i) If $m = 1$, then

$$\boldsymbol{\Phi} = \boldsymbol{\phi},\ \boldsymbol{\Theta} = \mathbf{0} \text{ and } \boldsymbol{\Omega}_E = \boldsymbol{\Omega}.$$

(ii) If $m > 1$, then

$$\boldsymbol{\Phi} = \boldsymbol{\phi}^m,$$

and $\boldsymbol{\Theta}$ will be the solution of the following quadratic matrix equation

$$\boldsymbol{\Theta}^2\boldsymbol{\Gamma}_1 + \boldsymbol{\Theta}\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1' = \mathbf{0},$$

where $\Gamma_0 = \mathrm{Var}(\boldsymbol{W}_{mT})$ and $\Gamma_1 = \mathrm{Cov}(\boldsymbol{W}_{mT}, \boldsymbol{W}_{mT+m})$, $\boldsymbol{W}_{mT} = (\boldsymbol{I} - \boldsymbol{\Phi}\mathrm{B})\boldsymbol{X}_T$, and

$$\Omega_E = -\Gamma_1(\Theta')^{-1}.$$

Proof :

(*i*) If $m = 1$, then $X_T = x_t$. Consequently,

$$\Phi = \phi\,,\ \Theta = \mathbf{0}\ \text{and}\ \Omega_E = \Omega. \tag{2.12}$$

(*ii*) If $m > 1$, by multiplying $(1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I-\phi^m\mathrm{B}^m)(I-\phi\mathrm{B})^{-1}$ on both sides of (2.10), we will obtain

$$(1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I-\phi^m\mathrm{B}^m)x_t = (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I-\phi^m\mathrm{B}^m)(I-\phi\mathrm{B})^{-1}a_t.$$

By changing t to mT and letting

$$W_{mT} = (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I-\phi^m\mathrm{B}^m)x_{mT} = (I-\phi^m\mathrm{B})X_T = (I-\Phi\mathrm{B})X_T$$

where $\Phi = \phi^m$, we note that

$$\begin{aligned} W_{mT} &= (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I-\phi^m\mathrm{B}^m)(I-\phi\mathrm{B})^{-1}a_{mT} \\ &= (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})(I+\phi\mathrm{B}+\ldots+\phi^{m-1}\mathrm{B}^{m-1})a_{mT} \\ &= \left[\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}\phi^i\right)\mathrm{B}^j + \sum_{j=1}^{m-1}\phi^j\left(\sum_{i=0}^{m-1-j}\phi^i\right)\mathrm{B}^{m-1+j}\right]a_{mT}. \end{aligned}$$

Therefore, the variance of W_{mT} is obtained as

$$\Gamma_0 = \mathrm{Var}(W_{mT}) = \left[\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}\phi^i\right)\Omega\left(\sum_{i=0}^{j}\phi^i\right)' + \sum_{j=1}^{m-1}\phi^j\left(\sum_{i=0}^{m-1-j}\phi^i\right)\Omega\left(\sum_{i=0}^{m-1-j}\phi^i\right)'(\phi^j)'\right]. \tag{2.13}$$

The autocovariance matrix is then obtained as

$$\Gamma_1 = \mathrm{Cov}(W_{mT}, W_{mT+m}) = \mathrm{E}(W_{mT}W'_{mT+m}),$$

$$\Gamma_1 = E\left\{\left\{\left[\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}\phi^i\right)B^j + \sum_{j=1}^{m-1}\phi^j\left(\sum_{i=0}^{m-1-j}\phi^i\right)B^{m-1+j}\right]a_{mT+m}\right\}\times\right.$$

$$\left.\times\left\{\left[\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}\phi^i\right)B^j + \sum_{j=1}^{m-1}\phi^j\left(\sum_{i=0}^{m-1-j}\phi^i\right)B^{m-1+j}\right]a_{mT}\right\}'\right\}.$$

Hence,

$$\Gamma_1 = \sum_{j=0}^{m-2}\left(\sum_{i=0}^{j}\phi^i\right)\Omega\left(\sum_{i=0}^{m-2-j}\phi^i\right)'\left(\phi^{j+1}\right)', \tag{2.14}$$

and

$$\Gamma_s = \mathrm{Cov}(\boldsymbol{W}_{mT}, \boldsymbol{W}_{mT+ms}) = \mathbf{0}, s > 1.$$

Thus, $\{\boldsymbol{W}_{mT}\} = \{(\boldsymbol{I} - \Phi B)\boldsymbol{X}_T\}$ is a vector MA(1) process. As a result, Θ is the solution of the following quadratic matrix equation

$$\Theta^2\Gamma_1 + \Theta\Gamma_0 + \Gamma_1' = \mathbf{0}, \tag{2.15}$$

where Γ_0 and Γ_1 are given in (2.13) and (2.14), respectively. After obtaining the moving average parameter matrix Θ, Ω_E is given as

$$\Omega_E = -\Gamma_1(\Theta')^{-1}. \tag{2.16}$$

These results show that if the basic time series follows a VAR(1) process

$$\boldsymbol{x}_t - \phi\boldsymbol{x}_{t-1} = \boldsymbol{a}_t, \tag{2.17}$$

the aggregate series will follow a VARMA(1,1) model.

∎

Proposition 2.3 Let $\boldsymbol{x}_t$ be a zero mean basic time series following a VARMA(0,1,1) process:

$$(1-\mathrm{B})\boldsymbol{x}_t = (\boldsymbol{I}+\Psi_1\mathrm{B})\boldsymbol{a}_t, \tag{2.18}$$

where $\boldsymbol{a}_t$ is a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix Ω. Then the aggregate time series defined by $\boldsymbol{X}_T = (1+\mathrm{B}+...+\mathrm{B}^{m-1})\boldsymbol{x}_{mT}$ will follow a VARMA (0,1,1) process

$$(1-\mathrm{B})\boldsymbol{X}_T = (\boldsymbol{I}-\Theta\mathrm{B})\boldsymbol{E}_T, \tag{2.19}$$

where $\boldsymbol{E}_T$ is a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix Ω_E. The moving average parameter Θ and the covariance matrix Ω_E are determined as

(i) If $m = 1$, then

$$\Theta = -\Psi_1 \text{ and } \Omega_E = \Omega.$$

(ii) If $m > 1$, then Θ will be the solution of the following quadratic matrix equation

$$\Theta^2\Gamma_1 + \Theta\Gamma_0 + \Gamma_1' = \mathbf{0},$$

where $\Gamma_0 = \mathrm{Var}(\Delta\boldsymbol{X}_T)$, $\Gamma_1 = \mathrm{Cov}(\Delta\boldsymbol{X}_T, \Delta\boldsymbol{X}_{T+1})$, $\Delta\boldsymbol{X}_T = (1-\mathrm{B})\boldsymbol{X}_T$, and $\Omega_E = -\Gamma_1(\Theta')^{-1}$.

Proof:

(i) If $m = 1$, then $\boldsymbol{X}_T = \boldsymbol{x}_t$. Consequently,

$$\Theta = -\Psi_1 \text{ and } \Omega_E = \Omega. \tag{2.20}$$

(ii) If $m > 1$, by multiplying $(1+B+...+B^{m-1})(1-B^m)$ on both sides of (2.18) we will obtain

$$(1+\mathrm{B}+...+\mathrm{B}^{m-1})(1-\mathrm{B}^m)(1-\mathrm{B})\boldsymbol{x}_t = (1+\mathrm{B}+...+\mathrm{B}^{m-1})(1-\mathrm{B}^m)(\boldsymbol{I}+\Psi_1\mathrm{B})\boldsymbol{a}_t.$$

Then,

$$(1+B+...+B^{m-1})(1-B^{m})\boldsymbol{x}_t = \frac{(1+B+...+B^{m-1})(1-B^{m})}{(1-B)}(\boldsymbol{I}+\Psi_1 B)\boldsymbol{a}_t.$$

Changing t to mT and letting $\Delta\boldsymbol{X}_T = (1+B+...+B^{m-1})(1-B^{m})\boldsymbol{x}_{mT} = (1-B)\boldsymbol{X}_T$,

$$\Delta\boldsymbol{X}_T = (1+B+...+B^{m-1})^2(\boldsymbol{I}+\Psi_1 B)\boldsymbol{a}_{mT}.$$

Ultimately, we have

$$\Delta\boldsymbol{X}_T = \left[\sum_{i=0}^{m-1}((i+1)\boldsymbol{I}_k + i\Psi_1)B^{i} + \sum_{i=1}^{m}((m-i)\boldsymbol{I}_k + (m-i+1)\Psi_1)B^{m+i-1}\right]\boldsymbol{a}_{mT}, \tag{2.21}$$

and so, therefore,

$$\begin{aligned}\Gamma_0 &= \mathrm{Var}(\Delta\boldsymbol{X}_T)\\ &= E\left[\left(\sum_{i=0}^{m-1}((i+1)\boldsymbol{I}_k + i\Psi_1)\Omega((i+1)\boldsymbol{I}_k + i\Psi_1)' + \sum_{i=1}^{m}((m-i)\boldsymbol{I}_k + (m-i+1)\Psi_1)\Omega((m-i)\boldsymbol{I}_k + (m-i+1)\Psi_1)'\right)\right],\end{aligned}$$

$$\Gamma_0 = \frac{m(2m^2+1)}{3}\{\Omega + \Psi_1\Omega\Psi_1'\} + \frac{2m(m^2-1)}{3}\{\Psi_1\Omega + \Omega\Psi_1'\}, \tag{2.22}$$

and

$$\begin{aligned}\Gamma_1 &= \mathrm{Cov}(\Delta\boldsymbol{X}_T, \Delta\boldsymbol{X}_{T+1}) = E(\Delta\boldsymbol{X}_T\Delta\boldsymbol{X}_{T+1}') =\\ &= \Omega(m-1) + m\Omega\Psi_1' + \sum_{i=1}^{m-2}((i+1)\boldsymbol{I}_k + i\Psi_1)\Omega((m-i-1)\boldsymbol{I}_k + (m-i)\Psi_1') + (m\boldsymbol{I}_k + (m-1)\Psi_1)\Omega\Psi_1',\end{aligned}$$

$$\Gamma_1 = \frac{m(m^2-1)}{6}\{\Omega + \Psi_1\Omega\Psi_1'\} + \frac{m(m-1)(m-2)}{6}\Psi_1\Omega + \frac{m(m+1)(m+2)}{6}\Omega\Psi_1', \tag{2.23}$$

$$\Gamma_s = \mathrm{Cov}(\Delta\boldsymbol{X}_T, \Delta\boldsymbol{X}_{T+s}) = \boldsymbol{0}, s > 1.$$

Thus, $\{\Delta\boldsymbol{X}_T\}$ is a vector MA(1) process. Hence, the aggregate process $\{\boldsymbol{X}_T\}$ is a vector ARIMA (0,1,1) model

$$(1-B)\boldsymbol{X}_T = \boldsymbol{E}_T - \boldsymbol{\Theta}\boldsymbol{E}_{T-1}, \tag{2.24}$$

with $\boldsymbol{E}_T$ having mean vector $\boldsymbol{0}$ and the covariance matrix Ω_E.

To determine $\boldsymbol{\Theta}$ we note that

$$\Gamma_0 = \mathrm{Var}(\Delta \boldsymbol{X}_T) = \boldsymbol{\Omega}_E + \boldsymbol{\Theta}\boldsymbol{\Omega}_E\boldsymbol{\Theta}',$$

$$\Gamma_1 = \mathrm{Cov}(\Delta \boldsymbol{X}_T, \Delta \boldsymbol{X}_{T+1}) = -\boldsymbol{\Omega}_E\boldsymbol{\Theta}'. \tag{2.25}$$

Hence, $\boldsymbol{\Theta}$ will be the solution of the following quadratic matrix equation

$$\boldsymbol{\Theta}^2\boldsymbol{\Gamma}_1 + \boldsymbol{\Theta}\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1' = \mathbf{0}, \tag{2.26}$$

where $\boldsymbol{\Gamma}_0$ and $\boldsymbol{\Gamma}_1$ are computed from (2.22) and (2.23). Once we obtain $\boldsymbol{\Theta}$, from (2.25), we will get $\boldsymbol{\Omega}_E = -\Gamma_1(\boldsymbol{\Theta}')^{-1}$.

Thus, when the process generating mechanism of the basic time series is a VARMA (0,1,1) model, the model for aggregate data remains the same but with different parameters. The unit root remains after temporal aggregation; it means that the aggregate time series is also non-stationary.

Proposition 2.4 Let $\boldsymbol{x}_t$ be a zero mean basic time series following a VAR(p) process:

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t, \tag{2.27}$$

where $\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\phi}_1\boldsymbol{B} - \cdots - \boldsymbol{\phi}_p\mathrm{B}^p$ and $\boldsymbol{a}_t$ are a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}$. Then the aggregate time series defined by $\boldsymbol{X}_T = (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})\boldsymbol{x}_{mT}$ will follow a VARMA (P, Q) process

$$\boldsymbol{\Phi}_p(\mathrm{B})\boldsymbol{X}_T = \boldsymbol{\Theta}_Q(\mathrm{B})\boldsymbol{E}_T, \tag{2.28}$$

where $\boldsymbol{\Phi}_p(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\Phi}_1\mathrm{B} - \cdots - \boldsymbol{\Phi}_p\mathrm{B}^p$, $\boldsymbol{\Theta}_Q(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\Theta}_1\mathrm{B} - \cdots - \boldsymbol{\Theta}_Q\mathrm{B}^Q$ with $Q = (p+1)(1-m^{-1})$ and $\boldsymbol{E}_T$ are a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}_E$.

Proof:

The matrix polynomial $\phi_p(\mathrm{B})$ can be written as

$$\phi_p(\mathrm{B}) = \boldsymbol{I} - \phi_1\mathrm{B} - \cdots - \phi_p\mathrm{B}^p = \prod_{i=1}^{p}(\boldsymbol{I} - \delta_i\mathrm{B}) \tag{2.29}$$

for δ_j's satisfying

$$\phi_j = (-1)^{j-1}\sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\cdots\sum_{i_{j-1}=i_{j-2}+1}^{p-1}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{j-1}}\boldsymbol{B}^{j-1},\ \phi_i = (-1)^{i-1}\sum_{j=1}^{p}\prod_{\ell=1}^{p-1}\delta_{j_\ell}, i = 1,\cdots,p-1 \text{ and}$$

$\phi_p = (-1)^p\prod_{j=1}^{p}\delta_j$ because

$$\prod_{i=1}^{p}(\boldsymbol{I} - \delta_i\mathrm{B}) = \boldsymbol{I} - \sum_{i=1}^{p}\delta_i\mathrm{B} + \sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\delta_{i_1}\delta_{i_2}\mathrm{B}^2 + \cdots +$$
$$+(-1)^{p-1}\sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\cdots\sum_{i_{p-1}=i_{p-2}+1}^{p-1}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{p-1}}\mathrm{B}^{p-1} + (-1)^p\prod_{i=1}^{p}\delta_i\mathrm{B}^p.$$

Then, equation (2.28) can be written as

$$\prod_{i=1}^{p}(\boldsymbol{I} - \delta_i\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t. \tag{2.30}$$

When we multiply both sides of equation (2.30) by

$$\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}(\boldsymbol{I} - \delta_i^m\mathrm{B}^m)(\boldsymbol{I} - \delta_i\mathrm{B})^{-1},$$

we get

$$\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}(\boldsymbol{I} - \delta_i^m\mathrm{B}^m)(\boldsymbol{I} - \delta_i\mathrm{B})^{-1}(\boldsymbol{I} - \delta_i\mathrm{B})\boldsymbol{x}_t = \prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}(\boldsymbol{I} - \delta_i^m\mathrm{B}^m)(\boldsymbol{I} - \delta_i\mathrm{B})^{-1}\boldsymbol{a}_t,$$

that is,

$$\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\boldsymbol{x}_t=\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\left(\boldsymbol{I}-\delta_i\mathrm{B}\right)^{-1}\boldsymbol{a}_t. \quad (2.31)$$

By changing t to mT,

$$\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\boldsymbol{x}_{mT}=\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\left(\boldsymbol{I}-\delta_i\mathrm{B}\right)^{-1}\boldsymbol{a}_{mT}. \quad (2.32)$$

Since $\boldsymbol{X}_T=\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\boldsymbol{x}_{mT}$, we will rewrite equation (2.32) as

$$\prod_{i=1}^{p}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\boldsymbol{X}_T=\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\left(\boldsymbol{I}-\delta_i\mathrm{B}\right)^{-1}\boldsymbol{a}_{mT}$$

or

$$\boldsymbol{\Phi}_p(\mathrm{B})\boldsymbol{X}_T=\prod_{i=1}^{p}\frac{(1-\mathrm{B}^m)}{(1-\mathrm{B})}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}^m\right)\left(\boldsymbol{I}-\delta_i\mathrm{B}\right)^{-1}\boldsymbol{a}_{mT}, \quad (2.33)$$

where $\boldsymbol{\Phi}_p(\mathrm{B})=\prod_{i=1}^{p}\left(\boldsymbol{I}-\delta_i^m\mathrm{B}\right)=\boldsymbol{I}-\boldsymbol{\Phi}_1\mathrm{B}-\cdots-\boldsymbol{\Phi}_p\mathrm{B}^p$. When we look at the right side of equation (2.33), we can see the order of the MA parameters. So, $Q=(m-1+p(m-1))/m=(p+1)\left(1-m^{-1}\right)$. Therefore, equation (2.33) is given by

$$\boldsymbol{\Phi}_p(\mathrm{B})\boldsymbol{X}_T=\boldsymbol{\Theta}_Q\boldsymbol{E}_T \quad (2.34)$$

where $\boldsymbol{\Phi}_p(\mathrm{B})=\boldsymbol{I}-\boldsymbol{\Phi}_1\mathrm{B}-\cdots-\boldsymbol{\Phi}_p\mathrm{B}^p$, $\boldsymbol{\Theta}_Q(\mathrm{B})=\boldsymbol{I}-\boldsymbol{\Theta}_1\mathrm{B}-\cdots-\boldsymbol{\Theta}_Q\mathrm{B}^Q$ with $Q=(p+1)\left(1-m^{-1}\right)$. In equation (2.34), all the parameters are functions of ϕ_i's and $\boldsymbol{\Omega}$.

CHAPTER 3

THE EFFECT OF TEMPORAL AGGREGATION ON COINTEGRATION

3.1 Introduction

The main idea of this chapter is to develop a new test statistic and its limiting distribution for cointegration under temporal aggregation. We will also demonstrate that cointegration in the system is not affected by aggregation. What is more, we obtain the vector error correction representation of an aggregated series for the vector autoregressive process of order 1. Then, based on this representation and following the work of Johansen (1988), we will develop a new test statistic and its limiting distribution to test cointegration in an aggregated model.

3.2 Temporal Aggregation of a Cointegration System

A *k*-dimensional vector time series $\boldsymbol{x}_t$ is said to be cointegrated of order *d*, *b*, denoted as $\boldsymbol{x}_t \sim CI(d,b)$, if all the components of $\boldsymbol{x}_t$ are *I(d),* and a linear combination of these component is *I(d-b), b*>0. When *d* and *b* values are $d=1$ and $b=1$ for a cointegrated system, their linear combination is $I(0)$, which is a stationary process. If there are more than two components in $\boldsymbol{x}_t$, then there may be more than one linearly independent cointegrating vectors. In this case, we let $\underset{h\times k}{A'}=\left[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_h\right]'$ which is a $h\times k$ cointegrating matrix composed of *h* linearly independent cointegrating vectors, and $A'\boldsymbol{x}_t$ is stationary.

A time series variable can be a flow variable or a stock variable. A series of a flow variable is often obtained through aggregation over equal time intervals such as partial sums, that is, $X = \{X_T\}_{T=0}^{\infty} = \left\{ \sum_{i=0}^{m-1} x_{mT-i} \right\}_{T=0}^{\infty}$. A series of a stock variable is often obtained by systematic sampling such that only the m^{th} elements of the original process are observed, that is, $X = \{X_T\}_{T=0}^{\infty} = \{x_{mT}\}_{T=0}^{\infty}$. It is natural to study the effect of aggregation for these flow and stock variables.

Marcellino (1999) proves that for a stock variable, when basic series are cointegrated, its temporal aggregates are also cointegrated. We prove in the following Theorem 3.1 that this result also holds true for a flow variable.

Theorem 3.1: Let x_t be a k dimensional vector series that is integrated with order 1, i.e. $I(1)$, and the process is cointegrated with a cointegrating matrix A of rank *h*, *i.e.* $A'x_t$ *is* $I(0)$. Then, the aggregate series X_T is also integrated with order 1. Furthermore, $A'X_T$ is also $I(0)$.

Proof:

Since x_t is $I(1)$, by definition,

$$\Delta x_t = \Psi(\mathrm{B}) a_t \tag{3.1}$$

is $I(0)$, where $\Psi(\mathrm{B}) = \sum_{j=0}^{\infty} \Psi_j \mathrm{B}^j$ such that $\sum_{j=0}^{\infty} |\Psi_j| < \infty$, and $\Psi(1) \neq 0$. The Wold representation of the aggregate series can be found by multiplying both sides of (3.1) by $(1 + B + ... + B^{m-1})(1 - B^m)/(1 - B)$

$$(1+B+...+B^{m-1})(1-B^m)\boldsymbol{x}_t = \frac{(1+B+...+B^{m-1})(1-B^m)}{1-B}\Psi(B)\boldsymbol{a}_t. \quad (3.2)$$

By changing t in (3.2) to mT

$$(1+B+...+B^{m-1})(1-B^m)\boldsymbol{x}_{mT} = (1+B+...+B^{m-1})^2\Psi(B)\boldsymbol{a}_{mT}.$$

$$(1-B)\boldsymbol{X}_T = \left(\sum_{i=0}^{m-1}(i+1)\boldsymbol{I}_k B^i + \sum_{i=0}^{m-1}(m-i-1)\boldsymbol{I}_k B^{m+i}\right)\left(\boldsymbol{I}+\Psi_1 B+\Psi_2 B^2+\cdots\right)\boldsymbol{a}_{mT} \quad (3.3)$$

The Wold representation of the aggregate series is then obtained as

$$(1-B)\boldsymbol{X}_T = \psi(B)\boldsymbol{E}_T \quad (3.4)$$

where $\psi(B)\boldsymbol{E}_T = \sum_{i=0}^{\infty}\psi_i \boldsymbol{E}_{T-i}$. Note that the ψ_i is absolutely summable since the Ψ_j is absolutely summable and ψ_i is a finite linear combination of the Ψ_j. Moreover, $\psi(1) \neq \boldsymbol{0}$, since $\left(\sum_{i=0}^{m-1}(i+1)\boldsymbol{I}_k B^i + \sum_{i=0}^{m-1}(m-i-1)\boldsymbol{I}_k B^{m+i}\right)\left(\boldsymbol{I}+\Psi_1 B+\Psi_2 B^2+\cdots\right)$ does not equal $\boldsymbol{0}$ at B =1. Hence, $(1-B)\boldsymbol{X}_T$ is $I(0)$, and $\boldsymbol{X}_T$ is $I(1)$.

Now $\boldsymbol{A}'\boldsymbol{x}_t$ is $I(0)$, hence it can be written in a form, $\boldsymbol{A}'\boldsymbol{x}_t = \sum_{i=0}^{\infty}\Upsilon_i \boldsymbol{a}_t$, where $\boldsymbol{A}'\boldsymbol{x}_t$ is stationary and $\Upsilon = \sum_{i=0}^{\infty}\Upsilon_i \neq \boldsymbol{0}$. For the aggregates, we note that

$$\boldsymbol{A}'\boldsymbol{X}_T = \boldsymbol{A}'\left(1+B+\cdots+B^{m-1}\right)\boldsymbol{x}_{mT} = \boldsymbol{A}'\boldsymbol{x}_{mT} + \boldsymbol{A}'\boldsymbol{x}_{mT-1} + \cdots + \boldsymbol{A}'\boldsymbol{x}_{mT-m+1}. \quad (3.5)$$

Clearly $\boldsymbol{A}'\boldsymbol{X}_T$ is I(0) since it is a finite sum of $I(0)$.

Q.E.D.

Given a k-dimensional cointegrated $I(1)$ series $\boldsymbol{x}_t$ that follows a vector autoregressive VAR(p) process:

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t. \tag{3.6}$$

where $\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\phi}_1\mathrm{B} - \ldots - \boldsymbol{\phi}_p\mathrm{B}^p$ and $\boldsymbol{\phi}_i$'s are $k \times k$ matrices, $\boldsymbol{a}_t$'s are i.i.d. k-dimensional random vectors with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Omega}$. Since $(1-\mathrm{B})\boldsymbol{x}_t$ is $I(0)$, we can write its Wold representation as

$$(1-\mathrm{B})\boldsymbol{x}_t = \Psi_1(\mathrm{B})\boldsymbol{a}_t, \tag{3.7}$$

where $\Psi_1(\mathrm{B}) = \left[\frac{\boldsymbol{\phi}_p(\mathrm{B})}{(1-\mathrm{B})}\right]^{-1}$. Assume that $A'\boldsymbol{x}_t$ is $I(0)$ and $rank(A) = h$. By Granger representation theorem given in Chapter 1, its error correction model (ECM) is given by

$$\Delta\boldsymbol{x}_t = \sum_{i=1}^{p-1}\Gamma_i\Delta\boldsymbol{x}_{t-i} + \Pi\boldsymbol{x}_{t-p} + \boldsymbol{a}_t, \tag{3.8}$$

where $\Gamma_i = -\boldsymbol{I} + \boldsymbol{\phi}_1 + \cdots + \boldsymbol{\phi}_i$, $i = 1,\ldots,p-1$, $\Pi = \boldsymbol{\phi}_p(1) = -\gamma A'$ where $\boldsymbol{A}$, the cointegrating matrix, and γ the adjustment coefficients, are $k \times h$ matrices. As introduced in Chapter 1, the most commonly used cointegration test is the test due to Johansen (1988, 1991) and Johansen and Juselius (1990) to test the hypothesis $H_0 : Rank(\Pi) = h$ versus the alternative greater than h, where h < k. Since $\Pi = \gamma A'$, this is equivalent to test that $\boldsymbol{A}$ and γ are of full column rank h, the number of independent cointegrating vectors that forms the matrix $\boldsymbol{A}$. The test based on the likelihood ratio leads to the test statistic which is the trace of a diagonal matrix of generalized eigenvalues from Π, i.e.,

$$-2\ln\Lambda = -\mathrm{n}\sum_{i=h+1}^{k}\ln(1\text{-}\hat{\lambda}_i), \tag{3.9}$$

where $\hat{\lambda}_i$ denote the eigenvalues such that $\hat{\lambda}_1 > \ldots > \hat{\lambda}_k > 0$. If the test statistic is greater than the critical value for rank h, then the null hypothesis that the cointegration rank equal to h is rejected.

The statistic $-2\ln\Lambda$ has the following limiting distribution which can be expressed in terms of a $(k-h)$ – dimensional Brownian motion $\boldsymbol{Y}$ as

$$\operatorname{tr}\left\{\int_0^1\left[(\mathrm{d}\boldsymbol{Y})\boldsymbol{Y}'\right]\left[\int_0^1 \boldsymbol{Y}\boldsymbol{Y}'dt\right]^{-1}\int_0^1\left[\boldsymbol{Y}(\mathrm{d}\boldsymbol{Y})'\right]\right\}. \tag{3.10}$$

Table 3.1 presents the percentiles of the asymptotic distribution for the trace obtained through a simulation and are tabulated in Johansen (1988).

Table 3.1 The Critical Values of the Trace Test

h	2.5%	5%	10%	50%	90%	95%	97.5%
1	0.0	0.0	0.0	0.6	2.9	4.2	5.3
2	1.6	1.9	2.5	5.4	10.3	12.0	13.9
3	7.0	7.8	8.8	14.0	21.2	23.8	26.1
4	16.0	17.4	19.2	26.3	35.6	38.6	41.2
5	28.3	30.4	32.8	42.1	53.6	57.2	60.3

Let $\boldsymbol{X}_T = (1+\mathrm{B}+...+\mathrm{B}^{\mathrm{m-1}})\boldsymbol{x}_{mT}$. Since $(1-\mathrm{B})\boldsymbol{x}_t$ is $I(0)$, by Theorem 3.1, $(1-\mathrm{B})\boldsymbol{X}_T$ is also $I(0)$. Thus,

$$(1-\mathrm{B})\boldsymbol{X}_T = \Psi(\mathrm{B})\boldsymbol{E}_T = \sum_{i=0}^{\infty}\Psi_i\boldsymbol{E}_{T-i}, \tag{3.11}$$

where $\sum_i|\Psi_i| < \infty$ and $\Psi(1) \neq \mathbf{0}$, and can be obtained through the relationship

$$\Psi(\mathrm{B})\boldsymbol{E}_T = \left(1+\mathrm{B}+\cdots+\mathrm{B}^{m-1}\right)^2\left[\frac{\phi_p(\mathrm{B})}{(1-\mathrm{B})}\right]^{-1}\boldsymbol{a}_{mT} \tag{3.12}$$

using the method introduced in Chapter 2.

Following the Granger Representation Theorem *ii)* given in Chapter 1, we have

$$\boldsymbol{\Phi}_P(\mathrm{B})\boldsymbol{X}_T = \Theta_Q(\mathrm{B})\boldsymbol{E}_T \tag{3.13}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_P(\mathrm{B}) &= \mathrm{adj}(\Psi(\mathrm{B}))/(1-\mathrm{B})^{k-1} \\ &= \boldsymbol{I} - \boldsymbol{\Phi}_1\mathrm{B} - \ldots - \boldsymbol{\Phi}_P\mathrm{B}^P\end{aligned}$$

and $\Theta_Q(\mathrm{B}) = \det\left(\Psi(\mathrm{B})\right)/(1-\mathrm{B})^h = 1 - \Theta_1 B - \ldots - \Theta_Q B^Q$.

In addition, since the cointegrating matrix $\boldsymbol{A}$ remains unchanged as shown in Theorem 3.1, using the Granger Representation Theorem *iv),* we obtain the following error correction model for the aggregates

$$\Delta \boldsymbol{X}_T = \sum_{i=1}^{P-1} \boldsymbol{\eta}_i \Delta \boldsymbol{X}_{T-i} + \Pi_{AG}\boldsymbol{X}_{T-P} + \Theta_Q(\mathrm{B})\boldsymbol{E}_T, \tag{3.14}$$

where $\boldsymbol{\eta}_i = -\boldsymbol{I} + \boldsymbol{\Phi}_1 + \cdots + \boldsymbol{\Phi}_i$, $i = 1, \ldots, P-1$, $\Pi_{AG} = -\boldsymbol{\Phi}_P(1) = \boldsymbol{\alpha}\boldsymbol{A}'$ for some $\boldsymbol{\alpha}$.

Example 3.1 Consider the following cointegrated VAR(1) process:

$$(\boldsymbol{I} - \phi\mathrm{B})\boldsymbol{x}_t = \begin{bmatrix} 1\text{-B} & 0 \\ -0.4\mathrm{B} & 1 \end{bmatrix}\boldsymbol{x}_t = \boldsymbol{a}_t, \tag{3.15}$$

where a_t is white noise with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$.

It can be easily shown that the system is cointegrated with the cointegrating rank of $h = 1$. By (3.7), we can also write (3.15) as

$$(1\text{-B})\boldsymbol{x}_t = \begin{bmatrix} 1 & 0 \\ -\dfrac{0.4\mathrm{B}}{1\text{-B}} & \dfrac{1}{1\text{-B}} \end{bmatrix}^{-1} \boldsymbol{a}_t,$$

which is equal to

$$(1\text{-B})\boldsymbol{x}_t = (\boldsymbol{I}+\Psi_1\text{B})\boldsymbol{a}_t = \left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0.4 & -1\end{bmatrix}\text{B}\right\}\boldsymbol{a}_t,$$

where $\Psi_1 = \begin{bmatrix}0 & 0\\ 0.4 & -1\end{bmatrix}$. Following the Granger representation theorem *iv)* and equation (3.8), the ECM of equation (3.15) can be obtained as

$$\Delta\boldsymbol{x}_t = \Pi\boldsymbol{x}_{t-1} + \boldsymbol{a}_t$$

or

$$\Delta\boldsymbol{x}_t = \gamma\boldsymbol{A}'\boldsymbol{x}_{t-1} + \boldsymbol{a}_t$$

where a cointegrating vector, $\boldsymbol{A}' = [-0.4 \quad 1]$ and $\gamma' = [0 \quad -1]'$. By Proposition 2.3, the aggregate series has the following form

$$(1-\text{B})\boldsymbol{X}_T = (\boldsymbol{I}-\Theta_1\text{B})\boldsymbol{E}_T, \tag{3.16}$$

where $\boldsymbol{E}_T$ is a sequence of random variables with mean vector $\boldsymbol{0}$ and the covariance matrix Ω_E. If m > 1, then $\Theta_1 = \begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{bmatrix}$ will be the solution of the following quadratic matrix equation

$$\Theta_1^2\Gamma_1 + \Theta_1\Gamma_0 + \Gamma_1' = \boldsymbol{0},$$

where

$$\Gamma_0 = \begin{bmatrix}\gamma_{11,0} & \gamma_{12,0}\\ \gamma_{21,0} & \gamma_{22,0}\end{bmatrix} = \frac{\text{m}(2\text{m}^2+1)}{3}\{\Omega+\Psi_1\Omega\Psi_1'\} + \frac{2\text{m}(\text{m}^2\text{-}1)}{3}\{\Psi_1\Omega+\Omega\Psi_1'\},$$

$$\Gamma_1 = \begin{bmatrix}\gamma_{11,1} & \gamma_{12,1}\\ \gamma_{21,1} & \gamma_{22,1}\end{bmatrix} = \frac{\text{m}(\text{m}^2\text{-}1)}{6}\{\Omega+\Psi_1\Omega\Psi_1'\} + \frac{\text{m}(\text{m-}1)(\text{m-}2)}{6}\Psi_1\Omega + \frac{\text{m}(\text{m+}1)(\text{m+}2)}{6}\Omega\Psi_1'.$$

This means that Θ_1 can be calculated by the following nonlinear equations.

$$\Theta_{11}^2\gamma_{11,1}+\Theta_{12}\Theta_{21}\gamma_{11,1}+\Theta_{12}(\Theta_{11}+\Theta_{22})\gamma_{21,1}+\Theta_{11}\gamma_{11,0}+\Theta_{12}\gamma_{12,0}+\gamma_{11,1}=0$$

$$\Theta_{11}^2\gamma_{12,1}+\Theta_{12}\Theta_{21}\gamma_{12,1}+\Theta_{12}(\Theta_{11}+\Theta_{22})\gamma_{22,1}+\Theta_{11}\gamma_{12,0}+\Theta_{12}\gamma_{22,0}+\gamma_{21,1}=0$$

$$\Theta_{21}(\Theta_{11}+\Theta_{22})\gamma_{11,1}+\Theta_{22}^2\gamma_{21,1}+\Theta_{12}\Theta_{21}\gamma_{21,1}+\Theta_{21}\gamma_{11,0}+\Theta_{22}\gamma_{12,0}+\gamma_{12,1}=0$$

$$\Theta_{21}(\Theta_{11}+\Theta_{22})\gamma_{12,1}+\Theta_{22}^2\gamma_{22,1}+\Theta_{12}\Theta_{21}\gamma_{22,1}+\Theta_{21}\gamma_{12,0}+\Theta_{22}\gamma_{22,0}+\gamma_{22,1}=0.$$

We used Mathcad software to solve this problem by using Levenberg-Marquardt method. This is a quasi-Newton method. At each step, Mathcad estimates the first partial derivatives of the errors with respect to the variables to be solved to create a Jacobian matrix. Ordinarily, Mathcad can determine the next estimate to make by computing the Gauss-Newton step for each variable.

Once we obtain $\mathbf{\Theta}_1$, we will get $\mathbf{\Omega}_E=-\Gamma_1(\mathbf{\Theta}_1')^{-1}$.

For m = 3, we obtain the moving average parameter matrix as $\mathbf{\Theta}_1=\begin{bmatrix}-0.189 & -0.074\\ -0.476 & 0.971\end{bmatrix}$

and covariance matrix of aggregates is $\mathbf{\Omega}_E=\begin{pmatrix}18.09 & 7.84\\ 7.84 & 5.66\end{pmatrix}$. It follows that

$$(1\text{-B})\boldsymbol{X}_T=\begin{bmatrix}1+0.189\text{B} & 0.074\text{B}\\ 0.476\text{B} & 1-0.971\text{B}\end{bmatrix}\boldsymbol{E}_T, \tag{3.17}$$

where $\mathbf{\Theta}_1(\text{B})=\boldsymbol{I}-\mathbf{\Theta}_1\text{B}=\begin{bmatrix}1+0.189\text{B} & 0.074\text{B}\\ 0.476\text{B} & 1-0.971\text{B}\end{bmatrix}$. From Equation (3.13), we have

$$\det(\mathbf{\Theta}_1(B))/(1-B)=(1+0.218B)=\theta(\text{B}),$$

and

$$\mathbf{\Phi}_P(B)=Adj(\mathbf{\Theta}_1(B))=\begin{pmatrix}1-0.971B & -0.074B\\ -0.476B & 1+0.189B\end{pmatrix},$$

where the cointegrating rank of h=1. The aggregate series will then have the following representation:

$$\Phi_P(B)X_T = \Theta_Q(B)E_T$$

where $\Theta_Q(B) = \theta(B)I_k$. Hence by Equation (3.13) the VARMA representation of (3.17) is

$$\begin{pmatrix} 1-0.971B & -0.074B \\ -0.476B & 1+0.189B \end{pmatrix} X_T = (1+0.218B)E_T. \tag{3.18}$$

From Equation (3.14), the error correction representation of the equation in (3.18) is given by

$$\Delta X_T = \Pi_{AG} X_{T-1} + \Theta(B)E_T, \tag{3.19}$$

where $\Pi_{AG} = -\Phi_1(1) = \begin{pmatrix} -0.079 & 0.074 \\ 0.476 & -1.189 \end{pmatrix} = \alpha A'$, and $\theta(B) = (1+0.218B)$.

Note that $A' = [-0.4 \quad 1]$, we can obtain the adjustment vector as $\alpha = [0.074 \quad -1.189]'$.

Table 3.2 presents the error correction model parameters for an aggregated series for different aggregation periods. This table indicates that only the cointegration vector remains the same after aggregation and the VAR(1) process turns out to be a VARMA(1,1) process.

Table 3.2 The 2-dimensional ECM Parameter Estimates from an Aggregated Series

m	Adjustment Vector	Cointegrating Vector	θ	Error Variance Matrix
1	$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0	$\begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$
3	$\begin{bmatrix} 0.074 \\ -1.189 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.218	$\begin{pmatrix} 18.09 & 7.84 \\ 7.84 & 5.66 \end{pmatrix}$
4	$\begin{bmatrix} 0.119 \\ -1.190 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.236	$\begin{pmatrix} 41.58 & 17.64 \\ 17.64 & 10.50 \end{pmatrix}$
6	$\begin{bmatrix} 0.213 \\ -1.167 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.252	$\begin{pmatrix} 136.99 & 56.90 \\ 56.90 & 28.16 \end{pmatrix}$
8	$\begin{bmatrix} 0.308 \\ -1.135 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.259	$\begin{pmatrix} 320.91 & 132.00 \\ 132.00 & 60.35 \end{pmatrix}$
10	$\begin{bmatrix} 0.404 \\ -1.099 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.260	$\begin{pmatrix} 624.48 & 255.28 \\ 255.28 & 111.95 \end{pmatrix}$
12	$\begin{bmatrix} 0.500 \\ -1.062 \end{bmatrix}$	$\begin{bmatrix} -0.4 \\ 1 \end{bmatrix}$	0.261	$\begin{pmatrix} 1077. & 438.5 \\ 438.5 & 187.61 \end{pmatrix}$

Example 3.2 Consider the following 3-dimensional cointegrated VAR(1) process:

$$\boldsymbol{x}_t - \begin{bmatrix} 0 & 0.4 & 0 \\ 0 & 1 & 0 \\ 0 & 0.8 & 0 \end{bmatrix} \boldsymbol{x}_{t-1} = \boldsymbol{a}_t, \tag{3.20}$$

where $\boldsymbol{a}_t$ is white noise with mean vector $\boldsymbol{0}$ and the covariance matrix

$$\boldsymbol{\Omega} = \begin{bmatrix} 3 & 0.1 & 1 \\ 0.1 & 1.5 & 0.4 \\ 1 & 0.4 & 2 \end{bmatrix}.$$

This system is cointegrated with the cointegration rank of $h = 2$. (3.20) can also be written as

$$(1\text{-B})\begin{bmatrix} 1/(1\text{-B}) & -0.4\text{B}/(1\text{-B}) & 0 \\ 0 & 1 & 0 \\ 0 & -0.8\text{B}/(1\text{-B}) & 1/(1\text{-B}) \end{bmatrix} \boldsymbol{x}_t = \boldsymbol{a}_t .$$

The Wold representation can then be obtained as

$$(1\text{-B})\boldsymbol{x}_t = \begin{bmatrix} 1\text{-B} & 0.4\text{B} & 0 \\ 0 & 1 & 0 \\ 0 & 0.8\text{B} & (1\text{-B}) \end{bmatrix} \boldsymbol{a}_t ,$$

which is equal to

$$(1\text{-B})\boldsymbol{x}_t = (\boldsymbol{I} + \Theta_1 \text{B})\boldsymbol{a}_t = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0.4 & 0 \\ 0 & 0 & 0 \\ 0 & 0.8 & -1 \end{bmatrix} \text{B} \right\} \boldsymbol{a}_t , \tag{3.21}$$

where $\Theta_1 = \begin{bmatrix} -1 & 0.4 & 0 \\ 0 & 0 & 0 \\ 0 & 0.8 & -1 \end{bmatrix}$. Following the Granger representation theorem iv) and Equation (3.8), the ECM of equation (3.20) can be obtained as

$$\Delta \boldsymbol{x}_t = \gamma A' \boldsymbol{x}_{t-1} + \boldsymbol{a}_t$$

where $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the cointegrating matrix $A = \begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$.

By Proposition 2.3, the aggregate series has the following form

$$(1\text{-B})\boldsymbol{X}_T = \boldsymbol{E}_T - \Theta_1 \boldsymbol{E}_{T-1} , \tag{3.22}$$

where $\boldsymbol{E}_T$ is a sequence of random variables with mean vector **0** and the covariance matrix Ω_E. If $m > 1$, then Θ_1 will be the solution of the following quadratic matrix equation

$$\Theta_1^2\Gamma_1 + \Theta_1\Gamma_0 + \Gamma_1' = \mathbf{0},$$

where

$$\Gamma_0 = \frac{m(2m^2+1)}{3}\{\Omega + \Psi_1\Omega\Psi_1'\} + \frac{2m(m^2\text{-}1)}{3}\{\Psi_1\Omega + \Omega\Psi_1'\},$$

$$\Gamma_1 = \frac{m(m^2\text{-}1)}{6}\{\Omega + \Psi_1\Omega\Psi_1'\} + \frac{m(m\text{-}1)(m\text{-}2)}{6}\Psi_1\Omega + \frac{m(m+1)(m+2)}{6}\Omega\Psi_1'.$$

Once we obtain Θ_1, we will get $\Omega_E = -\Gamma_1(\Theta_1')^{-1}$.

The aggregate series will then have the following representation:

$$\Phi_P(B)X_T = \Theta_Q(B)E_T$$

where $\Phi_P(B) = \text{adj}(\Theta_1(B)) = I - \Phi_1 B$ and $\Theta_Q(B) = \det\left(\Theta_1(B)\right) = 1 - \theta B$.

From equation (3.14), the error correction representation of the equation in (3.20) is given by

$$\Delta X_T = \vartheta A' X_{T-1} + E_T - \Theta E_{T-1}, \tag{3.23}$$

where $\Theta = \theta I_k$.

Table 3.3 presents the error correction model parameters for an aggregated series for different aggregation periods. This table indicates that only the cointegration vector remains the same after aggregation and the VAR(1) process turns out to be a VARMA(1,1) process. Temporal aggregation clearly changes the error correction representation of the basic model.

Table 3.3 The 3-dimensional ECM Parameter Estimates from an Aggregated Series

m	Adjustment Matrix	Cointegrating Matrix	Θ	Error Variance Matrix
1	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$	0	$\begin{pmatrix} 3 & 0.1 & 1 \\ 0.1 & 1.5 & 0.4 \\ 1 & 0.4 & 2 \end{pmatrix}$
2	$\begin{bmatrix} 1.176 & -0.033 \\ -0.47 & -0.083 \\ 0 & 1.15 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$	0.15	$\begin{pmatrix} 6.519 & 1.995 & 2.16 \\ 1.995 & 8.698 & 1.2 \\ 2.16 & 1.2 & 4 \end{pmatrix}$
3	$\begin{bmatrix} 1.25 & -0.0733 \\ 0.147 & -0.186 \\ 0 & 1.1915 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$	0.1915	$\begin{pmatrix} 11.331 & 7.675 & 3.45 \\ 7.675 & 26.73 & 2.4 \\ 3.45 & 2.4 & 6 \end{pmatrix}$
5	$\begin{bmatrix} 1.343 & -0.163 \\ 0.324 & -0.408 \\ 0 & 1.214 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$	0.214	$\begin{pmatrix} 28.66 & 39.63 & 6.6 \\ 39.63 & 117.21 & 6.0 \\ 6.6 & 6.0 & 10.0 \end{pmatrix}$
6	$\begin{bmatrix} 1.383 & -0.206 \\ 0.413 & -0.521 \\ 0 & 1.22 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0.4 & 0.8 \\ 0 & -1 \end{bmatrix}$	0.22	$\begin{pmatrix} 42.69 & 69.8 & 8.9 \\ 69.8 & 200.84 & 8.4 \\ 8.9 & 8.9 & 12.0 \end{pmatrix}$

3.3 Effects of Aggregation on the Cointegration Test: An Illustrative Example

To see the effects of aggregation on the cointegration test, consider the following 2-dimensional cointegrated VAR(1) process:

$$(\boldsymbol{I} - \phi \mathrm{B})\mathbf{x}_t = \boldsymbol{a}_t \tag{3.24}$$

where $\phi = \begin{bmatrix} 1 & 0 \\ 0.4 & 0 \end{bmatrix}$, $\boldsymbol{a}_t$ are i.i.d. normal random vectors with mean $\mathbf{0}$ and the variance-covariance matrix $\Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$. It can be easily seen that the system is cointegrated with the cointegrating rank of $h = 1$ and a cointegrating vector $A' = [-0.4 \quad 1]$. We first generate 600 observations from this cointegrated system and obtain its m^{th} order aggregates for various m. We then apply Johansen's trace test for cointegration to these data sets using the critical values given in Johansen (1988) by using SAS software. The results are shown in Table 3.4. As m increases beyond 3, except m=12, the test indicates no cointegration; this contradicts the theoretical result that we have proved in Theorem 3.1. Therefore, the test statistic to test cointegration in the system has to be modified for aggregate data. In the next section, we will develop this modified test statistic for the aggregate series.

3.4 The Cointegration Test and Temporal Aggregation

3.4.1. Derivation of the Test Statistic:

Given the error correction model for the aggregates given in equation (3.14), let

$$Z_{0T} = [\Delta X_T]_{k\times 1} \quad , \quad Z_{1T} = (\Delta X'_{T-1}, \ldots, \Delta X'_{T-P+1})'_{k(P-1)\times 1} \quad , \quad Z_{PT} = [X_{T-P}]_{k\times 1} \quad ,$$

$\eta = (\eta_1, \ldots, \eta_{P-1})_{k\times k(P-1)}$, $\Theta_{k\times Qk} = (\Theta_1 I_k \quad \cdots \quad \Theta_Q I_k)$, and $E_{Qk\times 1} = (E'_{T-1} \quad \cdots \quad E'_{T-Q})'$. We have

$$Z_{0T} = \eta Z_{1T} + \alpha A' Z_{PT} + E_T - \Theta E. \tag{3.25}$$

Table 3.4 The Trace Test for Cointegration on a VAR(1) Process for Various Order *m* of Aggregation

H_0: Rank	H_1: Rank >	Eigenvalue	Trace	Critical Value
		$m = 1$		
0	0	0.5228	445.28	12.0
1	1	0.0036	2.16	4.2
		$m = 3$		
0	0	0.5244	151.75	12.0
1	1	0.0192	3.87	4.2
		$m = 4$		
0	0	0.5609	126.84	12.0
1	1	0.0278	4.20	4.2
		$m = 6$		
0	0	0.5431	81.89	12.0
1	1	0.0430	4.35	4.2
		$m = 8$		
0	0	0.5388	61.73	12.0
1	1	0.0586	4.46	4.2
		$m = 10$		
0	0	0.5287	49.50	12.0
1	1	0.0831	5.12	4.2
		$m = 12$		
0	0	0.5256	40.71	12.0
1	1	0.0816	4.17	4.2

We then define the product moment matrices as

$$\mathbf{M}_{ij} = \left[\mathrm{N}^{-1} \sum_{T=1}^{N} Z_{iT} Z'_{jT} \right], \text{ (i, j = 0, 1, P)}, \qquad (3.26)$$

and

$$\aleph_{iE} = \left[\mathrm{N}^{-1} \sum_{T=1}^{N} Z_{iT} E' \right]_{k \times Qk}, \text{ (i= 0, 1, P)}, \qquad (3.27)$$

and

$$\aleph_{EE} = \left[\mathrm{N}^{-1} \boldsymbol{EE}' \right]_{Qk \times Qk}; \tag{3.28}$$

and define

$$\Im_{ij} = \left[\mathrm{M}_{ij} - \mathrm{M}_{i1} \mathrm{M}_{11}^{-1} \mathrm{M}_{1j} \right]_{k \times k}, \ (\mathrm{i, j} = 0, \mathrm{P}) \tag{3.29}$$

and

$$\boldsymbol{H}_{iE} = \left[\aleph_{iE} - \mathrm{M}_{i1} \mathrm{M}_{11}^{-1} \aleph_{1E} \right]_{k \times Qk}, \ (\mathrm{i} = 0, \mathrm{P}) \tag{3.30}$$

and finally

$$\boldsymbol{H}_{EE} = \left[\aleph_{EE} - \aleph_{E1} \mathrm{M}_{11}^{-1} \aleph_{1E} \right]_{Qk \times Qk}. \tag{3.31}$$

The likelihood function is then obtained as

$$\ln \boldsymbol{L}(\alpha, A, \Theta, \Omega_E) \propto -\frac{\mathrm{N}}{2} \ln |\Omega_E| - \frac{1}{2} \mathrm{tr} \left\{ \Omega_E^{-1} \left(\sum_{T=1}^{N} \boldsymbol{E}_T \right) \left(\sum_{T=1}^{N} \boldsymbol{E}_T \right)' \right\}, \tag{3.32}$$

$$\ln \boldsymbol{L}(\alpha, A, \Theta, \Omega_E) \propto -\frac{\mathrm{N}}{2} \ln |\Omega_E| - \frac{1}{2} \mathrm{tr} \left\{ \Omega_E^{-1} \left(\sum_{T=1}^{N} \boldsymbol{Z}_{0T} - \eta \boldsymbol{Z}_{1T} - \alpha A' \boldsymbol{Z}_{PT} + \Theta \boldsymbol{E} \right) \left(\sum_{T=1}^{N} \boldsymbol{Z}_{0T} - \eta \boldsymbol{Z}_{1T} - \alpha A' \boldsymbol{Z}_{PT} + \Theta \boldsymbol{E} \right)' \right\}. \tag{3.33}$$

For fixed values of α and A, the maximum likelihood estimation consists of a regression of $\boldsymbol{Z}_{0T} - \alpha A' \boldsymbol{Z}_{PT} + \Theta \boldsymbol{E}$ on $\boldsymbol{Z}_{1T}$ giving the equations

$$\sum_{T=1}^{N} \boldsymbol{Z}_{0T} \boldsymbol{Z}_{1T}' = \eta \sum_{T=1}^{N} \boldsymbol{Z}_{1T} \boldsymbol{Z}_{1T}' + \alpha A' \sum_{T=1}^{N} \boldsymbol{Z}_{PT} \boldsymbol{Z}_{1T}' - \Theta \sum_{T=1}^{N} \boldsymbol{E} \boldsymbol{Z}_{1T}'$$

where $\sum_{T=1}^{N} \boldsymbol{E}_T \boldsymbol{Z}_{1T}' = \mathbf{0}$. By using the product moment matrices, this equation can be written as

$$\mathrm{M}_{01} = \eta \mathrm{M}_{11} + \alpha A' \mathrm{M}_{1P} - \Theta \aleph_{E1}$$

or

$$\eta = \mathrm{M}_{01}\mathrm{M}_{11}^{-1} - \alpha A'\mathrm{M}_{1P}\mathrm{M}_{11}^{-1} + \Theta\aleph_{E1}\mathrm{M}_{11}^{-1}. \tag{3.34}$$

The log-likelihood function is then given by

$$\ln L(\alpha, A, \Theta, \Omega_E) \propto -\frac{\mathrm{N}}{2}\ln|\Omega_E| -$$

$$\frac{1}{2}\mathrm{tr}\left\{\Omega_E^{-1}\left(\sum_{T=1}^{N}\left\{\left(Z_{0T} - \mathrm{M}_{01}\mathrm{M}_{11}^{-1}Z_{1T}\right) - \alpha A'\left(Z_{PT} - \mathrm{M}_{1P}\mathrm{M}_{11}^{-1}Z_{1T}\right) + \Theta\left(E - \aleph_{E1}\mathrm{M}_{11}^{-1}Z_{1T}\right)\right\}\right)\times\right.$$

$$\left.\times\left(\sum_{T=1}^{N}\left\{\left(Z_{0T} - \mathrm{M}_{01}\mathrm{M}_{11}^{-1}Z_{1T}\right) - \alpha A'\left(Z_{PT} - \mathrm{M}_{1P}\mathrm{M}_{11}^{-1}Z_{1T}\right) + \Theta\left(E - \aleph_{E1}\mathrm{M}_{11}^{-1}Z_{1T}\right)\right\}\right)'\right\}.$$

Let's define

$$R_{0T} = \left[Z_{0T} - \mathrm{M}_{01}\mathrm{M}_{11}^{-1}Z_{1T}\right]_{k\times k},$$

$$R_{PT} = \left[Z_{PT} - \mathrm{M}_{P1}\mathrm{M}_{11}^{-1}Z_{1T}\right]_{k\times k},$$

$$R_E = \left[E - \aleph_{E1}\mathrm{M}_{11}^{-1}Z_{1T}\right]_{Qk\times k},$$

and also define $\Im_{ij} = R_{ij}R_{ij}'$, $H_{iE} = R_{ij}R_E'$, (i, j = 0, P), and $H_{EE} = R_E R_E'$. So, we can write the log-likelihood function as

$$\ln L(\alpha, A, \Theta, \Omega_E) \propto -\frac{\mathrm{N}}{2}\ln|\Omega_E| -$$

$$-\frac{1}{2}\mathrm{tr}\left\{\Omega_E^{-1}\left(\sum_{T=1}^{N}\left\{R_{0T} - \alpha A'R_{PT} + \Theta R_E\right\}\right)\left(\sum_{T=1}^{N}\left\{R_{0T} - \alpha A'R_{PT} + \Theta R_E\right\}\right)'\right\}. \tag{3.35}$$

For fixed values of α and A, the MLE of $\Theta = \left(\Theta_1 I_k \quad \cdots \quad \Theta_Q I_k\right)$ is calculated as the following

$$\frac{\partial \ln L(\alpha, A, \Theta, \Omega_E)}{\partial \Theta} = H_{0E} - \alpha A'H_{PE} + \Theta H_{EE} = \mathbf{0},$$

$$\hat{\Theta}(\alpha, A) = \left(\alpha A'H_{PE} - H_{0E}\right)H_{EE}^{-1}.$$

Now equation (3.32) can be written as

$$\ln L(\alpha,A,\Theta,\Omega_E)\propto-\frac{N}{2}\ln|\Omega_E|-\frac{1}{2}\mathrm{tr}\left\{\Omega_E^{-1}\left(\sum_{T=1}^{N}\left\{R_{0T}-\alpha A'R_{PT}+(\alpha A'H_{PE}-H_{0E})H_{EE}^{-1}R_E\right\}\right)\times\right.$$
$$\left.\times\left(\sum_{T=1}^{N}\left\{R_{0T}-\alpha A'R_{PT}+(\alpha A'H_{PE}-H_{0E})H_{EE}^{-1}R_E\right\}\right)'\right\}.$$

For a fixed value of A, the MLE of α is calculated as the following

$$\frac{\partial\ln L(\alpha,A,\Theta,\Omega_E)}{\partial\alpha}=-\Im_{0P}A+H_{0E}H_{EE}^{-1}H_{EP}A+\alpha A'\Im_{PP}A-2\alpha A'H_{PE}H_{EE}^{-1}H_{EP}A+$$
$$+\alpha A'H_{PE}H_{EE}^{-1}H_{EE}H_{EE}^{-1}H_{EP}A=0$$

$$\hat{\alpha}(A)=\left(\Im_{0P}A-H_{0E}H_{EE}^{-1}H_{EP}A\right)\left[A'\left(\Im_{PP}-H_{PE}H_{EE}^{-1}H_{EP}\right)A\right]^{-1}. \tag{3.36}$$

Let's define $F_{0P}=H_{0E}H_{EE}^{-1}H_{EP}$ and $F_{PP}=H_{PE}H_{EE}^{-1}H_{EP}$. Then, (3.36) is written as

$$\hat{\alpha}(A)=\left(\Im_{0P}A-F_{0P}A\right)\left[A'\left(\Im_{PP}-F_{PP}\right)A\right]^{-1}. \tag{3.37}$$

Equation (3.35) is can then be rewritten as

$$\ln L(\alpha,A,\Theta,\Omega_E)\propto-\frac{N}{2}\ln|\Omega_E|-$$
$$-\frac{1}{2}\mathrm{tr}\left\{\Omega_E^{-1}\left(\sum_{T=1}^{N}\left[R_{0T}-\left\{(\Im_{0P}A-F_{0P}A)\left[A'(\Im_{PP}-F_{PP})A\right]^{-1}\right\}A'R_{PT}+\right.\right.\right.$$
$$\left.\left.+\left(\left\{(\Im_{0P}A-F_{0P}A)\left[A'(\Im_{PP}-F_{PP})A\right]^{-1}\right\}A'H_{PE}-H_{0E}\right)H_{EE}^{-1}R_E\right]\times\right.$$
$$\times\left\{\left(\sum_{T=1}^{N}\left[R_{0T}-\left\{(\Im_{0P}A-F_{0P}A)\left[A'(\Im_{PP}-F_{PP})A\right]^{-1}\right\}A'R_{PT}+\right.\right.\right.$$
$$\left.\left.\left.+\left(\left\{(\Im_{0P}A-F_{0P}A)\left[A'(\Im_{PP}-F_{PP})A\right]^{-1}\right\}A'H_{PE}-H_{0E}\right)H_{EE}^{-1}R_E\right]'\right\}.$$

For a fixed value of A, the MLE of Ω_E is calculated as the following

$$\frac{\partial \ln L(\alpha, A, \Theta, \Omega_E)}{\partial \Omega_E} = \frac{N}{2}\Omega_E^{-1} - \frac{1}{2}\Omega_E^{-1}\Big\{\Im_{00} - \Im_{0P}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) +$$
$$+H_{0E}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) - H_{0E}H_{EE}^{-1}H_{E0} -$$
$$-(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'\Im_{P0} +$$
$$+(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'\Im_{PP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) -$$
$$-(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) +$$
$$+(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{E0} +$$
$$+(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{E0} -$$
$$-(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) +$$
$$+(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{EE}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) -$$
$$-(\Im_{0P} - F_{0P})A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'H_{PE}H_{EE}^{-1}H_{EE}H_{EE}^{-1}H_{E0} -$$
$$-H_{0E}H_{EE}^{-1}H_{E0} + H_{0E}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) -$$
$$-H_{0E}H_{EE}^{-1}H_{EE}H_{EE}^{-1}H_{EP}A\left[A'(\Im_{PP} - F_{PP})A\right]^{-1}A'(\Im_{P0} - F_{P0}) +$$
$$+H_{0E}H_{EE}^{-1}H_{EE}H_{EE}^{-1}H_{E0}\Big\}$$

$$\hat{\Omega}_E(A) = (\Im_{00} - F_{00}) - [\Im_{0P} - F_{0P}]A\left(A'[\Im_{PP} - F_{PP}]A\right)^{-1}A'[\Im_{P0} + F_{P0}]. \qquad (3.38)$$

where $F_{00} = H_{0E}H_{EE}^{-1}H_{E0}$ and $F_{P0} = H_{PE}H_{EE}^{-1}H_{E0}$.

The likelihood function now becomes proportional to

$$\left|\hat{\Omega}_E(A)\right|^{-N/2},$$

and so, to maximize the likelihood function we have to minimize

$$\min\left|\Im_{00} - F_{00} - [\Im_{0P} - F_{0P}]A\left(A'[\Im_{PP} - F_{PP}]A\right)^{-1}A'[\Im_{P0} - F_{P0}]\right|,$$

whereby the minimization is over all $k \times h$ matrices A.

$$\left|(\Im_{00} - F_{00}) - [\Im_{0P} - F_{0P}]A\left(A'[\Im_{PP} - F_{PP}]A\right)^{-1}A'[\Im_{P0} - F_{P0}]\right|,$$
$$\left|\Im_{00} - F_{00}\right|\left|I - (\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]A\left(A'[\Im_{PP} - F_{PP}]A\right)^{-1}A'[\Im_{P0} - F_{P0}]\right|.$$

By using Lütkepohl (1991)

$$\left|\Im_{00} - F_{00}\right|\left|I - A'\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]A\left(A'\left[\Im_{PP} - F_{PP}\right]A\right)^{-1}\right|,$$

$$\left|\Im_{00} - F_{00}\right| \times$$
$$\times\left|\left(A'\left[\Im_{PP} - F_{PP}\right]A\right) - A'\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]A\right| / \left|A'\left[\Im_{PP} - F_{PP}\right]A\right|,$$

we will minimize

$$\left|\left(A'\left[\Im_{PP} - F_{PP}\right]A\right) - A'\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]A\right| / \left|A'\left[\Im_{PP} - F_{PP}\right]A\right|,$$

or

$$\left|A'\left\{\left(\Im_{PP} - F_{PP}\right) - \left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]\right\}A\right| / \left|A'\left[\Im_{PP} - F_{PP}\right]A\right|, \quad (3.39)$$

with respect to the matrix A.

Following Johansen (1995, Lemma A8, p. 224), Equation (3.39) is minimized by

$$\left|\lambda\left(\Im_{PP} - F_{PP}\right) - \left(\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]\right)\right| = 0, \quad (3.40)$$

which leads to the solution related to the ordered eigenvalues $1 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_k > 0$ of $\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]$ with respect to $\left(\Im_{PP} - F_{PP}\right)$.

Let T denote the diagonal matrix of ordered eigenvalues and V the matrix of the corresponding eigenvectors, then

$$\left(\Im_{PP} - F_{PP}\right)U\mathrm{T} = \left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]V, \quad (3.41)$$

where $U = (u_1, \ldots, u_k)$ is normalized such that

$$U'\left(\Im_{PP} - F_{PP}\right)U = I.$$

Now, we choose $A = U\varepsilon$ where ε is $k \times h$ matrix, then we minimize

$$\left|\varepsilon'U'\left(\Im_{PP}-F_{PP}\right)U\varepsilon-\varepsilon'U'\left[\Im_{P0}-F_{P0}\right]\left(\Im_{00}-F_{00}\right)^{-1}\left[\Im_{0P}-F_{0P}\right]U\varepsilon\right|/\left|\varepsilon'U'\left(\Im_{PP}-F_{PP}\right)U\varepsilon\right|,$$
$$\left|\varepsilon'\varepsilon-\varepsilon'\mathrm{T}\varepsilon\right|/\left|\varepsilon'\varepsilon\right|.$$

This can be achieved by choosing ε to be the first h unit vector or by choosing $\hat{A}$ to be the first h eigenvectors of $\left[\Im_{P0}-F_{P0}\right]\left(\Im_{00}-F_{00}\right)^{-1}\left[\Im_{0P}-F_{0P}\right]$ with respect to $\left(\Im_{PP}-F_{PP}\right)$.

In other words, the first h columns of U would be

$$\hat{A}=(\hat{u}_1,\ldots,\hat{u}_h). \tag{3.42}$$

Equation (3.37) is then given by

$$\hat{\alpha}=\left(\Im_{0P}\hat{A}-F_{0P}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}\right]^{-1}, \tag{3.43}$$

and $\hat{\Theta}=\left(\hat{\alpha}\hat{A}'H_{PE}-H_{0E}\right)H_{EE}^{-1}$ with

$$\hat{\Omega}_E=\left(\Im_{00}-F_{00}\right)-\left[\Im_{0P}-F_{0P}\right]\hat{A}\left(\hat{A}'\left[\Im_{PP}-F_{PP}\right]\hat{A}\right)^{-1}\hat{A}'\left[\Im_{P0}-F_{P0}\right], \tag{3.44}$$

because the vectors are normalized by the condition $\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}=I$. Therefore, the maximized likelihood is given as

$$\mathrm{L}_{\max}^{-2/N}=\left|\Im_{00}-F_{00}\right|\left|I-\hat{A}'\left[\Im_{P0}-F_{P0}\right]\left(\Im_{00}-F_{00}\right)^{-1}\left[\Im_{0P}-F_{0P}\right]\hat{A}\right|, \tag{3.45}$$

in using (3.41) and (3.42)

$$\left[\Im_{P0}-F_{P0}\right]\left(\Im_{00}-F_{00}\right)^{-1}\left[\Im_{0P}-F_{0P}\right]\hat{A}=\left(\Im_{PP}-F_{PP}\right)\hat{A}\mathrm{T},$$

and since $\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}=I$, we can write (3.45) as

$$\mathrm{L}_{\max}^{-2/N}=\left|\left(\Im_{00}-F_{00}\right)\right|\left|I_h-\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}\mathrm{T}_A\right|=\left|\left(\Im_{00}-F_{00}\right)\right|\left|I_h-\mathrm{T}_A\right|,$$

$$L_{max}^{-2/N} = \left|\left(\Im_{00} - F_{00}\right)\right| \prod_{i=1}^{h} (1-\hat{\lambda}_i) \tag{3.46}$$

where T_A denotes the diagonal matrix of the ordered eigenvalues and $\hat{A}$ is the matrix of the corresponding eigenvectors. Maximized likelihood function under the whole parameter space will then be

$$\underset{\text{all parameter space}}{L_{max}^{-2/N}} = \left|\left(\Im_{00} - F_{00}\right)\right| \prod_{i=1}^{k} (1-\hat{\lambda}_i).$$

Thus, the ratio of these two likelihood functions gives

$$\Lambda_{agg} = \frac{L_{max}^{-2/N}}{\underset{\text{all parameter space}}{L_{max}^{-2/N}}} = \frac{\left|\left(\Im_{00} - F_{00}\right)\right| \prod_{i=1}^{h} (1-\hat{\lambda}_i)}{\left|\left(\Im_{00} - F_{00}\right)\right| \prod_{i=1}^{k} (1-\hat{\lambda}_i)} = \frac{1}{\prod_{i=h+1}^{k} (1-\hat{\lambda}_i)}.$$

And so, the likelihood ratio test statistic of the test for the aggregates is

$$-2\ln \Lambda_{agg} = -N \sum_{i=h+1}^{k} \ln(1-\hat{\lambda}_i) \tag{3.47}$$

where $\hat{\lambda}_i$'s are ordered eigenvalues $1 > \hat{\lambda}_1 > ... > \hat{\lambda}_k > 0$ of $\left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]$ with respect to $\left(\Im_{PP} - F_{PP}\right)$. This means that $\hat{\lambda}_i$'s are the solution of $\left|\lambda\left(\Im_{PP} - F_{PP}\right) - \left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]\right| = 0$.

It is important to note the difference between our test statistic in equation (3.47) and the test statistic in (3.9) derived by Johansen (1988). The eigenvalues $\hat{\lambda}_i$'s in Johansen's statistic are the solution of $\left|\lambda\Im_{PP} - \Im_{P0}\left(\Im_{00}\right)^{-1}\Im_{0P}\right| = 0$. Temporal aggregation affects the error structure of the process. We need to add this effect into the test statistic in terms of product moment matrices of error terms, and hence the eigenvalues $\hat{\lambda}_i$'s should be obtained from the solution of $\left|\lambda(\Im_{PP}-F_{PP})-[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]\right|=0$ when aggregates are used in the test..

3.4.2 Asymptotic Properties of the Test Statistic:

Since x_t is an integration of order 1, X_T is also integrated of order 1. So, ΔX_T is stationary and the null hypothesis is satisfied for some α and A of full rank h. Hence, we can express ΔX_T in terms of the E_T's by its moving average representation,

$$\Delta X_T = \Psi(\mathrm{B})E_T = \sum_{i=0}^{\infty}\Psi_i E_{T-i}. \tag{3.48}$$

where

$$\Psi(\mathrm{B})E_T = \left(1+\mathrm{B}+\cdots+\mathrm{B}^{m-1}\right)^2\left[\frac{\phi_p(\mathrm{B})}{(1-\mathrm{B})}\right]^{-1} a_{mT}$$

for some exponentially decreasing coefficient Ψ_i. The null space for $\Psi(1)'=\sum_{i=0}^{\infty}\Psi_i'$ given by $\{\xi \mid \Psi(1)'\xi=0\}$ is exactly the range space of Π'_{AG}, that is, the space spanned by the columns in A and vice versa. We thus have the following representations:

$$\Pi_{AG}=\alpha A' \text{ and } \Psi(1)=\varphi\tau\delta', \tag{3.49}$$

where φ and δ are $k\times(k-h)$ matrices of full rank consisting of vectors orthogonal to the vectors in A and α; $\tau=(\delta'\wp\varphi)^{-1}$ is $(k-h)\times(k-h)$ full rank matrix with $\wp$, and is the derivative of $\Phi(z)$ for z=1, and $\varphi' A=\delta'\alpha=\mathbf{0}$ (Johansen, 1991).

We can represent X_T as

$$X_T = \sum_{j=0}^{T} \Delta X_{T-j} = (X_T - X_{T-1}) + (X_{T-1} - X_{T-2}) + \cdots + (X_1 - X_0)$$

where $X_0 = \mathbf{0}$. The covariance function of ΔX_T's is given by

$$\pi_{AG}(i) = \mathrm{Var}(\Delta X_T, \Delta X_{T+i})_{k\times k}, \tag{3.50}$$

and we define the matrices as

$$\sigma_{ij} = \pi_{AG}(i-j) = \mathrm{E}\left(\Delta X_{T-i} \Delta X'_{T-j}\right)_{k\times k};\ \mathrm{i, j = 0, 1, \ldots, P\text{-}1} \tag{3.51}$$

$$\sigma_{Pi} = \left[\sum_{j=P-i}^{\infty} \pi_{AG}(j)\right]_{k\times k} \quad ;\ \mathrm{i = 0, 1, \ldots, P\text{-}1} \tag{3.52}$$

and

$$\sigma_{PP} = \left[-\sum_{j=-\infty}^{\infty} |j| \pi_{AG}(j)\right]_{k\times k}, \tag{3.53}$$

and finally as

$$\pi_{AG} = \left[\sum_{j=-\infty}^{\infty} \pi_{AG}(j)\right]_{k\times k}. \tag{3.54}$$

And so, the following relations hold:

$$\pi_{AG}(i) = \sum_{j=0}^{\infty} \Psi_j \Omega_E \Psi'_{j+i},$$

$$\pi_{AG} = \sum_{j=0}^{\infty} \Psi_j \Omega_E \sum_{j=0}^{\infty} \Psi'_j = \Psi(1) \Omega_E \Psi(1)',$$

$$\mathrm{Var}(X_{T-P}) = \sum_{j=-T+P}^{T-P} (\mathrm{T\text{-}P\text{-}}|\mathrm{j}|) \pi_{AG}(j),$$

$$\mathrm{Cov}(\boldsymbol{X}_{T-P}, \Delta\boldsymbol{X}_{T-i}) = \sum_{j=P-i}^{T-i} \boldsymbol{\pi}_{AG}(j),$$

which shows that

$$\mathrm{Var}(\boldsymbol{X}_T / \sqrt{\mathrm{T}}) \to \sum_{i=-\infty}^{\infty} \boldsymbol{\pi}_{AG}(i) = \boldsymbol{\pi}_{AG},$$

and

$$\mathrm{Cov}(\boldsymbol{X}_{T-P}, \Delta\boldsymbol{X}_{T-i}) \to \sum_{j=P-i}^{\infty} \boldsymbol{\pi}_{AG}(j) = \boldsymbol{\sigma}_{Pi}.$$

The relation

$$\mathrm{Var}(\boldsymbol{A}'\boldsymbol{X}_{T-P}) = (\mathrm{N\text{-}P}) \sum_{j=-N+P}^{N-P} \boldsymbol{A}'\boldsymbol{\pi}_{AG}(j)\boldsymbol{A} - \sum_{j=-N+P}^{N-P} |\mathrm{j}| \boldsymbol{A}'\boldsymbol{\pi}_{AG}(j)\boldsymbol{A}$$

shows that

$$\mathrm{Var}(\boldsymbol{A}'\boldsymbol{X}_{T-P}) \to \boldsymbol{A}'\boldsymbol{\sigma}_{PP}\boldsymbol{A}, \tag{3.55}$$

and since $\boldsymbol{A}'\boldsymbol{\Psi}(\mathbf{1}) = \mathbf{0}$ implies that $\boldsymbol{A}'\boldsymbol{\pi}_{AG} = \mathbf{0}$, the first term disappears in the limit.

To be able to find the asymptotic distribution of the test statistic given in equation (3.47), we need to prove the following five lemmas. We will now address the asymptotic behavior of the product moment matrices that are given in equations (3.26)-(3.31).

Let **W** be a Brownian motion in k-dimensions with the covariance matrix $\boldsymbol{\Omega}_E$.

Lemma 3.1: If $E\left(\hbar_t^4\right)$ is finite where $\hbar_T = \sum_{s=0}^{\infty} -\{\Psi_{s+1} + \Psi_{s+2} + \ldots\}\boldsymbol{E}_{T-s}$, for $N \to \infty$, then

$$\text{a) } \mathrm{N}^{-1/2}\boldsymbol{X}_{[Nt]} \overset{w}{\to} \Psi(1)\boldsymbol{W}(t), \tag{3.56}$$

b) $M_{ij} \xrightarrow{a.s.} \sigma_{ij}$, i, j = 0, 1, …, P-1, (3.57)

c) $M_{Pi} \xrightarrow{a.s.} \Psi(1)\int_0^1 W(dW)'\Psi(1)' + \sigma_{Pi}$, i = 0, 1, …, P-1, (3.58)

d) $A'M_{PP}A \xrightarrow{a.s.} A'\sigma_{PP}A$, (3.59)

e) $N^{-1}M_{PP} \xrightarrow{w} \Psi(1)\left\{\int_0^1 [W(s)][W(s)]' \, ds\right\}\Psi(1)'$, (3.60)

f) $\aleph_{PE} \xrightarrow{w} \left[\Psi(1)\int_0^1 W(s)dW(s)'\right] u'_{1\times Qk}$, where $u' = \begin{pmatrix}1 & \cdots & 1\end{pmatrix}'_{1\times Qk}$, (3.61)

g) $A'\aleph_{PE} \xrightarrow{a.s.} \mathbf{0}_{1\times Qk}$, (3.62)

h) $\aleph_{0E} \xrightarrow{a.s.} -\begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix}\text{diag}(\Omega_E)_{Qk\times Qk}$, (3.63)

i) $\aleph_{EE} \xrightarrow{a.s.} \text{diag}(\Omega_E)_{Qk\times Qk}$. (3.64)

Proof of Lemma 3.1:

a) Let the k-dimensional standard Wiener process **W**(.) be a continuous-time process associating with time $s \in [0,1]$ and with the $(k\times 1)$ vector **W**(s) satisfying the following:

i) **W**(0)=0.

ii) For any time $0 \le s_1 < s_2 < \ldots < s_t \le 1$, the changes [**W**($s_2$)-**W**($s_1$)], [**W**($s_3$)-**W**($s_2$)],…, [**W**($s_t$)-**W**($s_{t-1}$)] are independent multivariate normal with [**W**(s)-**W**(r)]~N(0,(s-r)$\mathbf{I_k}$).

iii) For any given realization, **W**(s) is continuous in s with probability 1.

Suppose that $\{E_T\}_{T=1}^{\infty}$ is a k-dimensional i.i.d. vector process with mean **0** and the covariance matrix Ω_E and let

$$C_E(s)=\begin{cases} 0 & \text{for} \quad 0\le s<1/N \\ \dfrac{E_1}{\sqrt{N}} & \text{for } 1/N\le s<2/N \\ \dfrac{E_1+E_2}{\sqrt{N}} & \text{for } 2/N\le s<3/N \\ \vdots & \quad\vdots \\ \dfrac{E_1+E_2+\ldots+E_n}{\sqrt{N}} & \text{for} \quad s=1 \end{cases} \tag{3.65}$$

that is,

$$C_E(s)=N^{-1/2}(E_1+\ldots+E_{[Ns]})=N^{-1/2}\sum_{T=1}^{[Ns]}E_T \tag{3.66}$$

where [s] is the integer part of s. By applying the central limit theorem, we then obtain

$$C_E(s)=\frac{\sqrt{[Ns]}}{\sqrt{N}}.\frac{1}{\sqrt{[Ns]}}\sum_{T=1}^{[Ns]}E_T=\sqrt{s}.\frac{1}{\sqrt{[Ns]}}\sum_{T=1}^{[Ns]}E_T\xrightarrow{L}\sqrt{s}.N(0,\Omega_E)=N(0,s\Omega_E) \tag{3.67}$$

as $N\to\infty$, because the E_T are i.i.d. with **0** mean and the covariance matrix Ω_E

$$\sqrt{N}C_E(.)\xrightarrow{L}W(.). \tag{3.68}$$

The equation (3.48) can be written as

$$\begin{aligned} X_T &= X_{T-1}+E_T+\Psi_1E_{T-1}+\Psi_2E_{T-2}+\ldots.. \\ &=(X_{T-2}+E_{T-1}+\Psi_1E_{T-2}+\Psi_2E_{T-3}+\ldots..)+E_T+\Psi_1E_{T-1}+\Psi_2E_{T-2}+\ldots.. \\ &\vdots \\ &=X_0+E_T+(I+\Psi_1)E_{T-1}+\ldots+(I+\Psi_1+\ldots+\Psi_{T-1})E_1+(\Psi_1+\ldots+\Psi_T)E_0+\ldots \end{aligned}$$

$$X_T=X_0+\Psi(1)(E_1+E_2+\ldots+E_T)+\hbar_T-\hbar_0 \tag{3.69}$$

where $\Psi_{r,ij}$ is the i-th row and j-the is the column element of Ψ_r, $\sum_{r=0}^{\infty}r\,|\Psi_{r,ij}|<\infty$, $\Psi(1)=I+\Psi_1+\Psi_2+\ldots$ and $\hbar_T=\sum_{s=0}^{\infty}-\{\Psi_{s+1}+\Psi_{s+2}+\ldots\}E_{T-s}$.

Therefore, let $\mathbb{Z}_X(s) = N^{-3/2} X_{[Ns]}$

$$\sqrt{N}\mathbb{Z}_X(s) = N^{-1}\left(\sum_{T=1}^{[Ns]} \Psi(1) E_T + \hbar_{[Ns]} - \hbar_0\right). \tag{3.70}$$

where $X_0 = 0$. By using (3.70), we can obtain

$$\sqrt{N}\mathbb{Z}_X(.) \xrightarrow{p} \Psi(1)\sqrt{N} C_E(.) \xrightarrow{L} \Psi(1) W(.), \tag{3.71}$$

where $\Psi(1)W(s)$ is distributed as $N(\mathbf{0}, s\Psi(1)\Omega_E\Psi'(1))$ because defining $S_N(s) = N^{-1}\hbar_{[Ns]}$, $S_N(.) \xrightarrow{p} 0$. To see this, note that

$$\begin{aligned} P\left\{\sup_{s\in[0,1]} \left|S_N(s)\right| > \varepsilon\right\} &= P\left\{\left[\left|N^{-1}\hbar_1\right| > \varepsilon\right] \text{ or .... or } \left[\left|N^{-1}\hbar_N\right| > \varepsilon\right]\right\} \\ &\leq N.P\left\{\left[\left|N^{-1}\hbar_t\right| > \varepsilon\right]\right\} \\ &\leq N\frac{E\left\{N^{-1}\hbar_t\right\}^4}{\varepsilon^4} \\ &= \frac{E\left(\hbar_t^4\right)}{N^3\varepsilon^4} \end{aligned} \tag{3.72}$$

where the next-to-last line follows from Chebyshev's inequality. Since $E\left(\hbar_t^4\right)$ is finite, this probability goes to zero as $N \to \infty$, establishing that $S_N(.) \xrightarrow{p} 0$, as claimed (Hamilton, 1994).

b) $M_{ij} = N^{-1}\sum_{T=1}^{N} Z_{iT} Z'_{jT} = N^{-1}\sum_{i=1}^{N} \Delta X_{T-i}\Delta X'_{T-j}$; i, j = 0 ,1,..., P-1.

By using the law of large numbers,

$$M_{ij} = N^{-1}\sum_{T=1}^{N} Z_{iT} Z'_{jT} = N^{-1}\sum_{T=1}^{N} \Delta X_{T-i}\Delta X'_{T-j} \xrightarrow{a.s.} E\left(\Delta X_{T-i}\Delta X'_{T-j}\right) = \sigma_{ij}.$$

c) $\mathrm{M}_{Pj} = \mathrm{N}^{-1}\sum_{T=1}^{N} \boldsymbol{Z}_{PT}\boldsymbol{Z}'_{jT} = \mathrm{N}^{-1}\sum_{T=1}^{N} \boldsymbol{X}_{T-P}\Delta\boldsymbol{X}'_{T-j}$, j = 0, 1,…, P-1.

By using (3.71),

$$\int_0^1 \sqrt{\mathrm{N}}\mathbb{Z}_x(\mathrm{s})\mathrm{ds} = \sum_{\mathrm{T}=1}^{\mathrm{N}} \mathrm{N}^{-1}\boldsymbol{X}_{[Ts]} \xrightarrow{w} \Psi(1)\int_0^1 \boldsymbol{W}(\mathrm{s})\mathrm{ds},$$

and since $\boldsymbol{X}_T = \sum_{j=0}^{T} \Delta\boldsymbol{X}_{T-j}$, $\Delta\boldsymbol{X}_{T-j} \xrightarrow{w} \Psi(1)\mathrm{d}(\boldsymbol{W}(\mathrm{s}))$.

Therefore,

$$\mathrm{M}_{Pi} = \mathrm{N}^{-1}\sum_{T=1}^{N} \boldsymbol{Z}_{PT}\boldsymbol{Z}'_{iT} = \mathrm{N}^{-1}\sum_{t=1}^{N} \boldsymbol{X}_{T-P}\Delta\boldsymbol{X}'_{T-i} = \mathrm{N}^{-1}\sum_{t=1}^{N}\left(\sum_{j=0}^{T-P} \Delta\boldsymbol{X}_j\right)\Delta\boldsymbol{X}'_{T-i}$$

$$\xrightarrow{w} \Psi(1)\int_0^1 \boldsymbol{W}(\mathrm{s})(\mathrm{d}(\boldsymbol{W}(\mathrm{s})))'\mathrm{ds}\Psi(1)' + \sum_{j=P-i}^{\infty} \mathrm{Var}(\Delta\boldsymbol{X}_t, \Delta\boldsymbol{X}'_{T+j})$$

$$= \Psi(1)\int_0^1 \boldsymbol{W}(\mathrm{s})(\mathrm{d}(\boldsymbol{W}(\mathrm{s})))'\mathrm{ds}\Psi(1)' + \boldsymbol{\sigma}_{Pi},$$

because $\mathrm{Cov}(\boldsymbol{X}_{T-P}, \Delta\boldsymbol{X}_{T-i}) \to \sum_{j=P-i}^{\infty} \pi_{AG}(j) = \boldsymbol{\sigma}_{Pi}$.

d) Given in (3.50).

e) By using (3.47), $\int_0^1 \sqrt{\mathrm{N}}\mathbb{Z}_X(\mathrm{s})\mathrm{ds} \xrightarrow{w} \Psi(1)\int_0^1 \boldsymbol{W}(\mathrm{s})\mathrm{ds}$.

Therefore,

$$N^{-1}\boldsymbol{M}_{pp} = \int_0^1 \left[\sqrt{\mathrm{N}}\mathbb{Z}_N(\boldsymbol{s})\right]\left[\sqrt{\mathrm{N}}\mathbb{Z}_N(\boldsymbol{s})\right]' \mathrm{ds} \xrightarrow{w} \Psi(1)\left\{\int_0^1 [\boldsymbol{W}(\mathrm{s})][\boldsymbol{W}(\mathrm{s})]' \mathrm{ds}\right\}\Psi(1)'.$$

f) Because

$$\sqrt{\mathrm{N}}\mathbb{Z}_X(s) = \mathrm{N}^{-1}\boldsymbol{X}_{[Ns]} = \mathrm{N}^{-1}\left(\sum_{T=1}^{[Ns]}\Psi(1)\boldsymbol{E}_T + \hbar_T - \hbar_0\right) \xrightarrow{L} \Psi(1)\boldsymbol{W}(s),$$

$\mathrm{N}^{-3/2}\boldsymbol{X}_{[Ns]} \xrightarrow{w} \Psi(1)\boldsymbol{W}(s)$ and $\mathrm{N}^{-1}\boldsymbol{E}'_{T-j} \xrightarrow{w} \mathrm{d}\boldsymbol{W}(s)$, j = 1,..., Q.

Hence,

$$\aleph_{PE} = \mathrm{N}^{-1}\sum_{T=1}^{N}\boldsymbol{X}_{T-P}\boldsymbol{E}' = \mathrm{N}^{-1}\left[\sum_{T=1}^{N}\boldsymbol{X}_{T-P}\boldsymbol{E}'_{T-1} \quad \cdots \quad \sum_{T=1}^{N}\boldsymbol{X}_{T-P}\boldsymbol{E}'_{T-Q}\right] \xrightarrow{w}$$

$$\left(\Psi(1)\int_0^1 \boldsymbol{W}(s)\mathrm{d}\boldsymbol{W}(s)' \quad \cdots \quad \Psi(1)\int_0^1 \boldsymbol{W}(s)\mathrm{d}\boldsymbol{W}(s)'\right) = \left[\Psi(1)\int_0^1 \boldsymbol{W}(s)\mathrm{d}\boldsymbol{W}(s)'\right]\boldsymbol{u}'_{1\times Qk},$$

where $\boldsymbol{u}' = \begin{pmatrix}1 & \cdots & 1\end{pmatrix}'_{1\times Qk}$ and is a unit vector of size Qk.

g)

$$\boldsymbol{A}'\aleph_{PE} = \mathrm{N}^{-1}\sum_{T=1}^{N}\boldsymbol{A}'\boldsymbol{X}_{T-P}\boldsymbol{E}' = \mathrm{N}^{-1}\sum_{T=1}^{N}\boldsymbol{A}'\left[\boldsymbol{X}_{-P} + \Psi(1)(\boldsymbol{E}_1 + \ldots + \boldsymbol{E}_{T-1}) + \hbar_{T-P} - \hbar_{-P}\right]\boldsymbol{E}'$$

$$= \mathrm{N}^{-1}\boldsymbol{A}'\left(\sum_{T=1}^{N}\left[\Psi(1)\sum_{i=1}^{T-P}\boldsymbol{E}_i + \hbar_{T-P} - \hbar_{-P}\right]\boldsymbol{E}'_{T-1} \quad \cdots \quad \sum_{T=1}^{N}\left[\Psi(1)\sum_{i=1}^{T-P}\boldsymbol{E}_i + \hbar_{T-P} - \hbar_{-P}\right]\boldsymbol{E}'_{T-Q}\right)$$

by (3.47) and assuming $\boldsymbol{X}_{-P} = 0$ and $\boldsymbol{A}'\Psi(1) = 0$, we will obtain

$$\boldsymbol{A}'\aleph_{PE} \xrightarrow{a.s.} \boldsymbol{0}_{1\times Qk},$$

where $\mathrm{N}^{-1}\sum_{T=1}^{N}\boldsymbol{E}_i\boldsymbol{E}'_i \xrightarrow{a.s.} \mathrm{diag}(\Omega_E)_{Qk\times Qk}$, i=1,...,Q.

h)

$$\aleph_{0E} = \mathrm{N}^{-1}\sum_{T=1}^{N}\Delta\boldsymbol{X}_T\boldsymbol{E}' = \mathrm{N}^{-1}\sum_{T=1}^{N}(\Pi_{AG}\boldsymbol{X}_{T-P} + \boldsymbol{E}_T - \sum_{i=1}^{Q}\Theta_i\boldsymbol{E}_i)\boldsymbol{E}'$$

$$= \mathrm{N}^{-1}\sum_{T=1}^{N}\left(\Pi_{AG}\left[\boldsymbol{X}_{-P} + \Psi(1)(\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots + \boldsymbol{E}_{T-P}) + \hbar_{T-P} - \hbar_{-P}\right] + \boldsymbol{E}_T - \sum_{i=1}^{Q}\Theta_i\boldsymbol{E}_i\right)\boldsymbol{E}'.$$

by (3.71) $\sum_{T=1}^{N}\left[\hbar_{T-P}-\hbar_{-P}\right]E' \xrightarrow{a.s.} \mathbf{0}$, assuming $X_{-P}=\mathbf{0}$, $\Pi_{AG}\Psi(1)=\mathbf{0}$. Also, by using the independence of errors $E_T E' \xrightarrow{a.s.} \mathbf{0}$. Since $\sum_{i=1}^{Q}\Theta_i E_i = \left(\Theta_1 I_k \quad \cdots \quad \Theta_Q I_k\right)\left(E'_{T-1} \quad \cdots \quad E'_{T-Q}\right)'$, we will obtain

$$\aleph_{0E} \xrightarrow{a.s.} -\left(\Theta_1 I_k \quad \cdots \quad \Theta_Q I_k\right)_{k\times Qk} \operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk},$$

where $N^{-1}\sum_{T=1}^{N} E_i E_i' \xrightarrow{a.s.} \operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}$, i = 1,…,Q.

i) By applying the strong law of large numbers, we will obtain

$$\aleph_{EE} = N^{-1}\sum_{T=1}^{N} EE' = N^{-1}\sum_{T=1}^{N}\begin{pmatrix} E_{T-1} \\ \vdots \\ E_{T-Q}\end{pmatrix}\left(E'_{T-1} \quad \cdots \quad E'_{T-Q}\right)$$

$$= N^{-1}\sum_{T=1}^{N}\begin{pmatrix} E_{T-1}E'_{T-1} & \cdots & E_{T-1}E'_{T-Q} \\ \vdots & \ddots & \vdots \\ E_{T-Q}E'_{T-1} & \cdots & E_{T-Q}E'_{T-Q}\end{pmatrix}$$

$$\xrightarrow{a.s.} \begin{pmatrix} \Omega_E & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \Omega_E \end{pmatrix} = \operatorname{diag}(\Omega_E)_{Qk\times Qk}$$

Q.E.D. (for Lemma 3.1)

Now let's introduce the following notations:

$$\Im_{ij} = \left[\mathbf{M}_{ij} - \mathbf{M}_{i*}\mathbf{M}_{**}^{-1}\mathbf{M}_{*j}\right]_{k\times k},\ \text{i, j} = 0, \text{P},$$

where $\mathbf{M}_{**} = \left[\mathbf{M}_{ij}\right]_{k\times k}$, i,j=1,…,P-1, $\mathbf{M}_{P*} = \left[\mathbf{M}_{Pi}\right]_{k\times k}$, i=1,…,P-1, $\mathbf{M}_{0*} = \left[\mathbf{M}_{0i}\right]_{k\times k}$, i=1,…,P-1.

and

$$\Sigma_{ij} = \left[\sigma_{ij} - \sigma_{i*}\sigma_{**}^{-1}\sigma_{*j}\right]_{k\times k},\ \text{i, j} = 0, \text{P}. \tag{3.73}$$

where $\sigma_{ij}, \sigma_{i*}, \sigma_{**}$ and σ_{*j} defined by the relation (3.51), (3.52) and (3.53).

Lemma 3.2: If Σ_{ij}, i, j = 0, P is defined as in equation (3.73), then

$$\Sigma_{00} = \Pi_{AG}\Sigma_{P0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E \tag{3.74}$$

$$\Sigma_{0P}\Pi'_{AG} = \Pi_{AG}\Sigma_{PP}\Pi'_{AG} - \Theta\Omega_E\Pi'_{AG}, \tag{3.75}$$

and since $\Pi_{AG} = \alpha A'$ and $\alpha A'\Psi(1) = \mathbf{0}$,

$$\Sigma_{00} = \alpha A'\Sigma_{PP}A\alpha' + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E \tag{3.76}$$

and

$$\alpha = \Sigma_{0P}A(A'\Sigma_{PP}A)^{-1}. \tag{3.77}$$

Proof of Lemma 3.2:

The ECM of the aggregated series can be written as

$$\Delta X_T = \sum_{i=1}^{P-1}\eta_i\Delta X_{T-i} + \Pi_{AG}X_{T-P} + E_T - \Theta E. \tag{3.78}$$

In multiplying both sides by ΔX_{T-i}, i = 0, 1,…, P-1 , we will obtain

$$\Delta X_T\Delta X'_{T-i} = \sum_{i=1}^{P-1}\eta_i\Delta X_{T-i}\Delta X'_{T-i} + \Pi_{AG}X_{T-P}\Delta X'_{T-i} + E_T\Delta X'_{T-i} - \sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-i}.$$

And then, we will obtain

$$\mathrm{N}^{-1}\sum_{T=1}^{N}\Delta X_T\Delta X'_{T-i}=\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{i=1}^{P-1}\eta_i\Delta X_{T-i}\Delta X'_{T-i}+$$

$$+\mathrm{N}^{-1}\sum_{T=1}^{N}\Pi_{AG}X_{T-P}\Delta X'_{T-i}+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-i}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-i},$$

that is,

$$\mathrm{M}_{0i}=\sum_{j=1}^{P-1}\eta_j\mathrm{M}_{ji}+\alpha A'\mathrm{M}_{Pi}+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-i}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-i},\ \text{for i = 0,..,P-1.}$$

(3.79)

Multiplying both sides of (3.78) by X'_{T-P} gives

$$\Delta X_T X'_{T-P}=\sum_{i=1}^{P-1}\eta_i\Delta X_{T-i}X'_{T-P}+\Pi_{AG}X_{T-P}X'_{T-P}+E_T X'_{T-P}-\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}.$$

We will then obtain

$$\mathrm{M}_{0P}=\sum_{j=1}^{P-1}\eta_j\mathrm{M}_{jP}+\alpha A'\mathrm{M}_{PP}+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}. \qquad (3.80)$$

Letting $N\to\infty$ gives the equations

$$\sigma_{00}=\sum_{j=1}^{P-1}\eta_j\sigma_{j0}+\alpha A'\sigma_{P0}+\Omega_E+\Theta_1^2\Omega_E+\ldots+\Theta_Q^2\Omega_E, \qquad (3.81)$$

$$\sigma_{0i}=\sum_{i=1}^{P-1}\eta_j\sigma_{ji}+\alpha A'\sigma_{Pi},\ \text{i = 1, 2,..., P-1} \qquad (3.82)$$

$$\sigma_{P0}A=\sum_{j=1}^{P-1}\eta_j\sigma_{jP}A+\alpha A'\sigma_{PP}A. \qquad (3.83)$$

Because $Q\le P$, $\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}$ goes to **0** as $N\to\infty$.

Defining $\eta_* = \left\{\eta_j, j\text{=}1,\ldots,\text{P-1}\right\}$, (3.81), (3.82), and (3.83) can be written as

$$\sigma_{00} = \eta_*\sigma_{*0} + \alpha A'\sigma_{P0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E, \quad (3.84)$$

$$\sigma_{0i} = \eta_*\sigma_{**} + \alpha A'\sigma_{Pi},\ \text{i} = 1, 2,\ldots, \text{P-1} \quad (3.85)$$

$$\sigma_{P0}A = \eta_*\sigma_{*P}A + \alpha A'\sigma_{PP}A. \quad (3.86)$$

By using (3.85),

$$\eta_* = \sigma_{0*}\sigma_{**}^{-1} - \alpha A'\sigma_{P*}\sigma_{**}^{-1}. \quad (3.87)$$

Replacing (3.85) in (3.84) and (3.86) gives us

$$\sigma_{00} = \sigma_{0*}\sigma_{**}^{-1}\sigma_{*0} - \alpha A'\sigma_{P*}\sigma_{**}^{-1}\sigma_{*0} + \alpha A'\sigma_{P0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E$$
$$\sigma_{00} - \sigma_{0*}\sigma_{**}^{-1}\sigma_{*0} = \alpha A'(\sigma_{P0} - \sigma_{P*}\sigma_{**}^{-1}\sigma_{*0}) + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E$$

$$\Sigma_{00} = \alpha A'\Sigma_{p0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E, \quad (3.88)$$

$$\Sigma_{00} = \Pi_{AG}\Sigma_{p0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E, \quad (3.89)$$

and

$$\sigma_{0p}A = \sigma_{0*}\sigma_{**}^{-1}\sigma_{P*}A - \alpha A'\sigma_{P*}\sigma_{**}^{-1}\sigma_{P*}A + \alpha A'\sigma_{PP}A,$$
$$(\sigma_{0P} - \sigma_{0*}\sigma_{**}^{-1}\sigma_{P*})A = \alpha A'(\sigma_{PP} - \sigma_{P*}\sigma_{**}^{-1}\sigma_{P*})A,$$

$$\Sigma_{0P}A = \alpha A'\Sigma_{PP}A, \quad (3.90)$$

where $\Pi_{AG} = \alpha A'$.

By multiplying both sides of (3.90) by α', we have

$$\Sigma_{0P}\Pi'_{AG} = \Pi_{AG}\Sigma_{PP}\Pi'_{AG}.$$

Also, by using (3.90), we obtain α as

$$\alpha = \Sigma_{0P}A\left(A'\Sigma_{PP}A\right)^{-1}. \quad (3.91)$$

Q.E.D. (for Lemma 3.2)

Lemma 3.3 If δ is chosen such that $\delta'\alpha = \mathbf{0}$, then, for $N \to \infty$,

$$\text{a) } \Im_{00} \xrightarrow{a.s.} \Sigma_{00}, \tag{3.92}$$

$$\text{b) } \delta\Im_{0P} \xrightarrow{w} \delta' \int_0^1 (dW)W'\Psi(1)' - \delta' \int_0^1 \left[\left(\sum_{i=1}^{Q} \Theta_i\right) dW\right] W'\Psi(1)', \tag{3.93}$$

$$\text{c) } A'\Im_{P0} \xrightarrow{a.s.} A'\Sigma_{P0}, \tag{3.94}$$

$$\text{d) } \mathrm{N}^{-1}\Im_{PP} \xrightarrow{w} \Psi(1) \int_0^1 W(s)W'(s)ds\Psi(1)', \tag{3.95}$$

$$\text{e) } A'\Im_{PP}A \xrightarrow{a.s.} A'\Sigma_{PP}A, \tag{3.96}$$

$$\text{f) } H_{PE} \xrightarrow{w} \left[\Psi(1) \int_0^1 W(s)\mathrm{d}W(s)'\right] u'_{1\times Qk}, \text{ where } u'_{1\times Qk} = \begin{pmatrix}1 & \cdots & 1\end{pmatrix}'_{1\times Qk}, \tag{3.97}$$

$$\text{g) } A'H_{PE} \xrightarrow{a.s.} \mathbf{0}_{1\times Qk}, \tag{3.98}$$

$$\text{h) } H_{EE} \xrightarrow{a.s.} \operatorname{diag}(\Omega_E)_{Qk\times Qk}, \tag{3.99}$$

$$\text{i) } H_{0E} \xrightarrow{a.s.} -\begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix} \operatorname{diag}(\Omega_E)_{Qk\times Qk}. \tag{3.100}$$

Proof of Lemma 3.3:

Equations (3.78) and (3.79) can be written as

$$\mathrm{M}_{0*} = \eta_*\mathrm{M}_{**} + \alpha A'\mathrm{M}_{P*} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T \Delta X'_{T-*} - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q} \Theta_j E_{T-j} \Delta X'_{T-*}, \tag{3.101}$$

and

$$\mathrm{M}_{0P} = \eta_*\mathrm{M}_{*P} + \alpha A'\mathrm{M}_{PP} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T X'_{T-P} - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q} \Theta_j E_{T-j} X'_{T-P}. \tag{3.102}$$

Solving (3.101) for η_* gives us

$$\eta_* = \mathrm{M}_{0*}\mathrm{M}_{**}^{-1} - \alpha A'\mathrm{M}_{P*}\mathrm{M}_{**}^{-1} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T \Delta X'_{T-*}\mathrm{M}_{**}^{-1} - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1} \quad (3.103)$$

a) Equation (3.78) when i = 0 can be written as

$$\mathrm{M}_{00} = \eta_*\mathrm{M}_{*0} + \alpha A'\mathrm{M}_{P*} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T\Delta X'_T - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_T . \qquad (3.104)$$

By replacing (3.103) with (3.104), we will obtain

$$\begin{aligned}\mathrm{M}_{00} = \mathrm{M}_{0*}\mathrm{M}_{**}^{-1} - \alpha A'\mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0} - \\ -\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0} + \alpha A'\mathrm{M}_{P*} + \\ +\mathrm{N}^{-1}\sum_{T=1}^{N} E_T\Delta X'_T - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_T,\end{aligned}$$

that is,

$$\begin{aligned}\Im_{00} = \alpha A'\left(\mathrm{M}_{P0} - \mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0}\right) + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0} - \\ -\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*0} + \mathrm{N}^{-1}\sum_{T=1}^{N} E_T\Delta X'_T - \mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_T.\end{aligned}$$

As $N \to \infty$, using Lemma 3.1, Lemma 3.2 equation (3.88) and

$$\mathrm{N}^{-1}\sum_{T=1}^{N}\Theta_j E_{T-j}\Delta X'_T \xrightarrow{a.s.} \Theta_j^2\Omega_E ,$$

$$\begin{aligned}\Im_{00} &\xrightarrow{a.s.} \alpha A'\left(\sigma_{P0} - \sigma_{P*}\sigma_{**}^{-1}\sigma_{*0}\right) + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E \\ &= \alpha A'\Sigma_{P0} + \Omega_E + \Theta_1^2\Omega_E + \ldots + \Theta_Q^2\Omega_E = \Sigma_{00}.\end{aligned}$$

b) Solving (3.101) for η_* and inserting into (3.102) gives us

$$\mathrm{M}_{0P}=\mathrm{M}_{0P}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-\alpha A'\mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-$$
$$-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+\alpha A'\mathrm{M}_{PP}+$$
$$+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}$$

that is,

$$\Im_{0P}=\alpha A'(\mathrm{M}_{PP}-\mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P})+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-$$
$$-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+ \qquad (3.105)$$
$$+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}.$$

Multiplying both sides of (3.105) by δ' gives us

$$\delta'\Im_{0P}=\delta'\alpha A'(\mathrm{M}_{PP}-\mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P})+\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-$$
$$-\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+ \qquad (3.106)$$
$$+\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}.$$

Since E_T and $\Delta X'_{T-*}$ are stationary and uncorrelated, $\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}\xrightarrow{N\to\infty}0$.

$\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}\xrightarrow{a.s.}0$, $\delta'\alpha A'\Im_{PP}=0$ because $\chi'\alpha=0$. Then, with

(3.72), $\mathrm{N}^{-1/2}X_{T-P}\xrightarrow{w}\Psi(1)W$, and with (3.71) we are given $\delta' E_T\xrightarrow{w}\delta' dW$.

Also,

$$\mathrm{N}^{-1}\delta'\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P} = \mathrm{N}^{-1}\delta'\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\left(X_{-P}+\Psi(1)\left(E_1+\ldots+E_{T-P}\right)+\hbar_{T-P}-\hbar_{-P}\right)$$

$$\xrightarrow{w}\delta'\int_0^1\left(\sum_{j=1}^{Q}\Theta_j dW\right)W'\Psi(1)'$$

Therefore,

$$\delta\Im_{0P}\xrightarrow{w}\delta'\int_0^1(dW)W'\Psi(1)'-\delta'\int_0^1\left[\left(\sum_{i=1}^{Q}\Theta_i\right)dW\right]W'\Psi(1)'.$$

c) Solving (3.101) for η_* and inserting into (3.102) gives us

$$\mathrm{M}_{0P}=\mathrm{M}_{0*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-\alpha A'\mathrm{M}_{P*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}-\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}+$$

$$+\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+\alpha A'\mathrm{M}_{PP}+$$

$$+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P},$$

that is,

$$\Im_{0P}=\alpha A'\Im_{PP}-\mathrm{N}^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}\mathrm{M}_{**}^{-1}+\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}\mathrm{M}_{**}^{-1}\mathrm{M}_{*P}+$$
$$+\mathrm{N}^{-1}\sum_{T=1}^{N}E_T X'_{T-P}-\mathrm{N}^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}X'_{T-P}, \tag{3.107}$$

As $N\to\infty$, from Lemma 3.1, the equation in (3.107) becomes

$$\Im_{0P}A\xrightarrow{a.s.}\alpha A'\Sigma_{PP}A. \tag{3.108}$$

Since E_T and $\Delta X'_{T-i}$ are stationary and uncorrelated, $n^{-1}\sum_{t=1}^{n}E_T\Delta X'_{T-i}M_{**}^{-1}M_{*p}\xrightarrow{n\to\infty}0$.

Moreover,

$$\mathrm{N}^{-1}\mathbf{M}_{P*} = \mathrm{N}^{-1}\sum_{T=1}^{N} Z_{PT}Z'_{*T} = \mathrm{N}^{-1}\sum_{T=1}^{N} X_{T-P}(\Delta X'_{T-*}) = \mathrm{N}^{-1}\sum_{T=1}^{N} X_{T-P}(X'_{T-*} - X'_{T-*-1})$$

$$= \mathrm{N}^{-1}\sum_{T=1}^{N}(X_{T-P}X'_{T-*} - X_{T-P}X'_{T-*-1}) = \mathrm{N}^{-1}\sum_{T=1}^{N} X_{T-P}X'_{T-*} - \mathrm{N}^{-1}\sum_{T=1}^{N} X_{T-P}X'_{T-*-1} \quad (3.109)$$

$$\xrightarrow{w} \Psi(1)\int_0^1 WW'\Psi(1)' - \Psi(1)\int_0^1 WW'\Psi(1)' = 0.$$

By using (3.77),

$$\Im_{0P}A \xrightarrow{a.s.} \Sigma_{0P}A(A'\Sigma_{PP}A)^{-1}A'\Sigma_{PP}A = \Xi_{0P}A,$$

or

$$A'\Im_{P0} \xrightarrow{a.s.} A'\Sigma_{P0}. \quad (3.110)$$

d) By the definition of $\Im_{PP}$ in terms of $\mathbf{M}$'s

$$\Im_{PP} = \mathbf{M}_{PP} - \mathbf{M}_{P*}\mathbf{M}_{**}^{-1}\mathbf{M}_{*P},$$

where $\mathbf{M}_{ij} = N^{-1}\sum_{T=1}^{N} Z_{iT}Z'_{jT}$ and $\mathbf{M}_{P*} = \sum_{T=1}^{N} Z_{PT}Z'_{*T}$, and where $Z'_{*T} = (\Delta X'_{T-i}, \mathrm{i}=1,\dots,\mathrm{P}-1)$.

By using Lemma 3.1, e)

$$\mathrm{N}^{-1}\mathbf{M}_{PP} \xrightarrow{w} \Psi(1)\left\{\int_0^1 [W(s)][W(s)]'\,\mathrm{ds}\right\}\Psi(1)'.$$

and (3.108), we have $\mathbf{M}_{P*}\mathbf{M}_{**}^{-1}\mathbf{M}_{*P} \xrightarrow{w} \mathbf{0}$, and

hence,

$$\mathrm{N}^{-1}\Im_{PP} \xrightarrow{w} \Psi(1)\left\{\int_0^1 [W(s)][W(s)]'\,\mathrm{ds}\right\}\Psi(1)'.$$

e) In multiplying both sides of (3.106) by A, we write

$$\Im_{0P}A=\alpha A'\Im_{PP}A-N^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}M_{**}^{-1}A+N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_jE_{T-j}\Delta X'_{T-*}M_{**}^{-1}M_{*P}A+$$
$$+N^{-1}\sum_{T=1}^{N}E_TX'_{T-P}A-N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_jE_{T-j}X'_{T-P}A,$$

or

$$\alpha A'\Im_{PP}A=\Im_{0P}A+N^{-1}\sum_{T=1}^{N}E_T\Delta X'_{T-*}M_{**}^{-1}A\text{-}N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_jE_{T-j}\Delta X'_{T-*}M_{**}^{-1}M_{*P}A-$$
$$\text{-}N^{-1}\sum_{T=1}^{N}E_TX'_{T-P}A+N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_jE_{T-j}X'_{T-P}A, \qquad (3.111)$$

As $N\to\infty$, by using (3.108), the equation in (3.111) changes to

$$\alpha A'\Im_{PP}A\xrightarrow{a.s.}\alpha A'\Sigma_{PP}A\,,$$

that is,

$$A'\Im_{PP}A\xrightarrow{a.s.}A'\Sigma_{PP}A\,.$$

f) By using Lemma 3.1 and $\aleph_{1P}\xrightarrow{a.s.}0$, we will obtain

$$H_{PE}=\aleph_{PE}-M_{P1}M_{11}^{-1}\aleph_{1P}\xrightarrow{w}\Psi(1)\int_0^1 W(s)dW(s)'\,. \qquad (3.112)$$

g) $A'H_{PE}=A'(\aleph_{PE}-M_{P1}M_{11}^{-1}\aleph_{1E})\xrightarrow{a.s.}0$ because of Lemma 3.1 and (3.109).

h) $H_{EE}=\aleph_{EE}-\aleph_{E1}M_{11}^{-1}\aleph_{1E}\xrightarrow{a.s.}\text{diag}\left(\Omega_E\right)_{Qk\times Qk}$ because $\aleph_{1E}\xrightarrow{a.s.}0$.

i) In multiplying both sides of (3.77) by $E'\Theta'=\sum_{i=1}^{Q}E'_{T-i}\Theta_i$, we obtain

$$\Delta X_TE'\Theta'=\sum_{i=1}^{P-1}\eta_i\Delta X_{T-i}E'\Theta'+\Pi_{AG}X_{T-P}E'\Theta'+E_TE'\Theta'-\Theta EE'\Theta'\,.$$

Therefore,

$$N^{-1}\sum_{T=1}^{N}\Delta X_T E'\Theta' = N^{-1}\sum_{T=1}^{N}\sum_{i=1}^{P-1}\eta_i \Delta X_{T-i}E'\Theta' + N^{-1}\sum_{T=1}^{N}\Pi_{AG}X_{T-P}E'\Theta' +$$
$$+N^{-1}\sum_{T=1}^{N}E_T E'\Theta' - N^{-1}\sum_{T=1}^{N}\Theta E E'\Theta'$$

or

$$\aleph_{0E}\Theta' = \eta_* \aleph_{*E}\Theta' + \Pi_{AG}\aleph_{PE}\Theta' + N^{-1}\sum_{T=1}^{N}E_T E'\Theta' - \Theta\aleph_{EE}\Theta'. \tag{3.113}$$

In replacing η_* by using (3.103), we have

$$\aleph_{0E}\Theta' = M_{0*}M_{**}^{-1}\aleph_{*E}\Theta' - \alpha A' M_{P*}M_{**}^{-1}\aleph_{*E}\Theta' +$$
$$+N^{-1}\sum_{T=1}^{N}E_T \Delta X'_{T-*}M_{**}^{-1}\aleph_{*E}\Theta' - N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}M_{**}^{-1}\aleph_{*E}\Theta' +$$
$$+\Pi_{AG}\aleph_{PE}\Theta' + N^{-1}\sum_{T=1}^{N}E_T E'\Theta' - \Theta\aleph_{EE}\Theta'$$

or

$$H_{0E}\Theta' = \alpha A' H_{PE}\Theta' + N^{-1}\sum_{T=1}^{N}E_T \Delta X'_{T-*}M_{**}^{-1}\aleph_{*E}\Theta' -$$
$$-N^{-1}\sum_{T=1}^{N}\sum_{j=1}^{Q}\Theta_j E_{T-j}\Delta X'_{T-*}M_{**}^{-1}\aleph_{*E}\Theta' + N^{-1}\sum_{T=1}^{N}E_T E'\Theta' - \Theta\aleph_{EE}\Theta'.$$

By using Lemma 3.1, (3.98), and (3.99)

$$H_{0E} \xrightarrow{a.s.} -\left(\Theta_1 I_k \quad \cdots \quad \Theta_Q I_k\right)\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}$$

Q.E.D. (for Lemma 3.3)

Hence, by (3.98), we can state that

$$A' F_{PP} \to \mathbf{0}, \tag{3.114}$$

$$A'F_{P0} \to \mathbf{0}, \tag{3.115}$$

and

$$F_{0P}A \to \mathbf{0}. \tag{3.116}$$

Moreover, by (3.99) and (3.100),

$$F_{00} \overset{a.s.}{\to} \begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix} \operatorname{diag}(\Omega_E)_{Qk\times Qk} \left(\operatorname{diag}(\Omega_E)_{Qk\times Qk}\right)^{-1} \operatorname{diag}(\Omega_E)_{Qk\times Qk} \begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix}'$$
$$= \begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix} \operatorname{diag}(\Omega_E)_{Qk\times Qk} \begin{pmatrix}\Theta_1 I_k & \cdots & \Theta_Q I_k\end{pmatrix}' = \sum_{i=1}^{Q} \Theta_i^2 \Omega_E. \tag{3.117}$$

Lemma 3.4: If $\lambda_1,\ldots,\lambda_h$ are the ordered eigenvalues of the equation

$$\left|\lambda A'\Sigma_{PP}A - A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}A\right| = 0 \tag{3.118}$$

where $\zeta = \Sigma_{00} - \sum_{i=1}^{Q}\Theta_i^2\Omega_E$, then the ordered eigenvalues of the equation

$$\left|\lambda\left(\Im_{PP} - F_{PP}\right) - \left[\Im_{P0} - F_{P0}\right]\left(\Im_{00} - F_{00}\right)^{-1}\left[\Im_{0P} - F_{0P}\right]\right| = 0 \tag{3.119}$$

converge in probability to $(\lambda_1,\ldots,\lambda_h,0,\ldots,0)$.

Proof of Lemma 3.4:

We express the problem in the coordinates given by k vectors in A and γ where $\Psi(1) = \gamma\wp\delta'$, $\wp$ is a $(k-h)\times(k-h)$, γ and δ are $k\times(k-h)$, all are of full rank, and $\gamma'A = \delta'\alpha = \mathbf{0}$. We will eventually choose δ in a more convenient way. This can be done by multiplying (3.119) by $|(A,\gamma G_N)'|$ and $|(A,\gamma G_N)|$ from the left and right where $G_N = (\gamma'\ (\Im_{PP} - F_{PP})\gamma)^{-1/2}$; then the eigenvalues solve the equation

$$\left|\lambda\begin{bmatrix} A'(\Im_{PP}-F_{PP})A & A'(\Im_{PP}-F_{PP})\gamma G_N \\ G_N'\gamma'(\Im_{PP}-F_{PP})A & G_N'\gamma'(\Im_{PP}-F_{PP})\gamma G_N \end{bmatrix} - \right.$$

$$\left. -\begin{bmatrix} A'[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]A & A'[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]\gamma G_N \\ G_N'\gamma'[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]A & G_N'\gamma'[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]\gamma G_N \end{bmatrix}\right| = 0$$

By using Lemma 3.3, $\gamma' A = 0$, $\Psi(1)' A = 0$, and

$G_N'\gamma'(\Im_{PP}-F_{PP})\gamma G_N = \left[(\gamma'(\Im_{PP}-F_{PP})\gamma)^{-1/2}\right]'\gamma'(\Im_{PP}-F_{PP})\gamma(\gamma'(\Im_{PP}-F_{PP})\gamma)^{-1/2} = I$;

the eigenvalues have to satisfy the following equation

$$\left|\lambda\begin{bmatrix} A'\Sigma_{PP}A & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}A & 0 \\ 0 & 0 \end{bmatrix}\right| = 0 \tag{3.120}$$

or

$$\left|\lambda A'\Sigma_{PP}A - A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}\right|\left|\lambda I_{k-h}\right| = 0 \tag{3.121}$$

where

$$\Im_{00} - F_{00} \overset{a.s.}{\rightarrow} \zeta = \Sigma_{00} - \sum_{i=1}^{Q}\Theta_i^2\Omega_E \tag{3.122}$$

and I_{k-h} is an identity matrix of the dimension k–h, which means that the equation has k–h roots at $\lambda = 0$. It is known that the ordered eigenvalues are continuous functions of the coefficient matrix (Anderson, Brons and Jensen, 1983), and hence the proof of Lemma 3.4 has been completed.

Q.E.D. (for Lemma 3.4)

Based on Lemma 3.4, we can say that

$$\hat{A} \overset{a.s.}{\rightarrow} A \tag{3.123}$$

where $\hat{A}$ is the first h eigenvectors of $[\Im_{P0}-F_{P0}](\Im_{00}-F_{00})^{-1}[\Im_{0P}-F_{0P}]$ with respect to $(\Im_{PP} - F_{PP})$. By using Lemma 3.3 and (3.91)

$$\hat{\alpha} = \left(\Im_{OP}\hat{A} - H_{0E}H_{EE}^{-1}H_{EP}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP} - H_{PE}H_{EE}^{-1}H_{EP}\right)\hat{A}\right]^{-1}$$
$$= \left(\Im_{OP}\hat{A} - F_{0P}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP} - F_{PP}\right)\hat{A}\right]^{-1}$$

$$\hat{\alpha} \xrightarrow{a.s.} \alpha = \Sigma_{0P}A(A'\Sigma_{PP}A)^{-1}. \tag{3.124}$$

We now choose δ of the dimension such that $\delta'\alpha = 0$ in the following way. Let $P_\alpha(\zeta)$ denote the projection of R^p onto the column space spanned by α with respect to matrix ζ^{-1} because $\Psi(1)'\hat{A} = 0$; that is,

$$P_\alpha(\zeta) = \alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha'\zeta^{-1}. \tag{3.125}$$

We can then choose δ of full rank k–h to satisfy

$$\delta\delta' = \zeta^{-1}(I - P_\alpha(\zeta)). \tag{3.126}$$

Note that $\delta'\alpha = 0$, therefore

$$\delta\delta' = \zeta^{-1}(I - P_\alpha(\zeta)) = \zeta^{-1}(I - \alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha'\zeta^{-1})$$
$$= \zeta^{-1} - \zeta^{-1}\alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha'\zeta^{-1} = (I - \zeta^{-1}\alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha')\zeta^{-1}$$
$$\delta\delta'\zeta = (I - \zeta^{-1}\alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha')$$
$$\delta\delta'\zeta\delta = (I - \zeta^{-1}\alpha(\alpha'\zeta^{-1}\alpha)^{-1}\alpha')\delta = \delta$$
$$\delta'\zeta\delta = I$$

of the dimension $(k-h)\times(k-h)$. From the theory of random coefficient regression (Rao, 1965), $P_\alpha(\Omega_E) = P_\alpha(\Sigma_{00})$ where Σ_{00} is given by using (3.74).

Lemma 3.5: For $N \to \infty$, $N\hat{\lambda}_{h+1}, \ldots, N\hat{\lambda}_k$ converge in distribution to the ordered eigenvalues of the equation

$$\left| \int_0^1 GG' \mathrm{d}u - \beta \left[\int_0^1 G (\mathrm{d}G)' \right] \left[\int_0^1 (\mathrm{d}G) G' \right] \right| = 0 \tag{3.127}$$

where G is a Brownian motion in the k–h dimension with the covariance matrix I.

Proof of Lemma 3.5:

Consider the ordered eigenvalues of the equation

$$\left| \begin{bmatrix} A'(\Im_{PP} - F_{PP})A/\mathrm{N} & A'(\Im_{PP} - F_{PP})\gamma/\mathrm{N} \\ \gamma'(\Im_{PP} - F_{PP})A/\mathrm{N} & \gamma'(\Im_{PP} - F_{PP})\gamma/\mathrm{N} \end{bmatrix} - \right.$$

$$\left. \beta \begin{bmatrix} A'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]A & A'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]\gamma \\ \gamma'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]A & \gamma'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]\gamma \end{bmatrix} \right| = 0$$

(3.128)

For any value of N, the ordered eigenvalues are

$$\hat{\beta}_1 = (N\hat{\lambda}_k)^{-1}, \ldots, \hat{\beta}_k = (N\hat{\lambda}_1)^{-1}.$$

Since the ordered eigenvalues are continuous functions of the coefficients, we know from Lemma 3.3 that $\hat{\beta}_1, \ldots, \hat{\beta}_k$ converge in distribution to the ordered eigenvalues of the equation

$$\beta \begin{bmatrix} A'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]A & A'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]\gamma \\ \gamma'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]A & \gamma'[\Im_{P0} - F_{P0}](\Im_{00} - F_{00})^{-1}[\Im_{0P} - F_{0P}]\gamma \end{bmatrix} \Bigg|$$

$$\left| \begin{bmatrix} 0 & 0 \\ 0 & \gamma' \Psi(1) \int_0^1 YY'\Psi(1)'\gamma \end{bmatrix} - \beta \begin{bmatrix} A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}A & A'\Sigma_{P0}\zeta^{-1}\left(\int_0^1 (\mathrm{d}Y)Y'\Psi(1)' \right)\gamma \\ \left(\Psi(1) \int_0^1 Y(\mathrm{d}Y)' \right)\zeta^{-1}\Sigma_{0P}A\gamma' & \gamma'\left(\Psi(1) \int_0^1 Y(\mathrm{d}Y)' \right)\zeta^{-1}\left(\int_0^1 (\mathrm{d}Y)Y'\Psi(1)' \right)\gamma \end{bmatrix} \right| = 0.$$

This determinant can also be written as

$$\left|\beta A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}\right| \cdot \left|\gamma'\,\Psi(1)\int_0^1 YY'\mathrm{d}u\,\Psi(1)'\gamma - \beta\gamma'\left[\Psi(1)\int_0^1 Y(\mathrm{d}Y)'\right]\times\right.$$
$$\left.\times\left\{\zeta^{-1}-\zeta^{-1}\Sigma_{0P}A\left[A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}A\right]^{-1}A'\Sigma_{P0}\zeta^{-1}\right\}\left[\int_0^1(\mathrm{d}Y)Y'\Psi(1)'\right]\gamma\right| \tag{3.129}$$

which shows that in the limit there are h roots at zero. By applying Lemma 3.2, (3.123), (3.124) and (3.125), we find

$$\zeta^{-1}-\zeta^{-1}\Sigma_{0P}A\left[A'\Sigma_{P0}\zeta^{-1}\Sigma_{0P}A\right]^{-1}A'\Sigma_{P0}\zeta^{-1}$$

equals

$$\zeta^{-1}(I-P_{\alpha}(\zeta))=\delta\delta', \tag{3.130}$$

and hence, the second factor of (3.129) is

$$\left|\gamma'\,\Psi(1)\int_0^1 YY'\mathrm{d}u\,\Psi(1)'\gamma-\beta\gamma'\left[\Psi(1)\int_0^1 Y(\mathrm{d}Y)'\right]\delta\delta'\left[\int_0^1(\mathrm{d}Y)Y'\Psi(1)'\right]\gamma\right|. \tag{3.131}$$

Thus, the limiting distribution of the k–h largest β's is given as that of the ordered eigenvalues for the equation in (3.131). And so, (3.131) can also be written as

$$\left|\gamma'\,\Psi(1)\right|\left|\int_0^1 YY'\mathrm{d}u-\beta\int_0^1 Y(\mathrm{d}Y)'\delta\delta'\int_0^1(\mathrm{d}Y)Y'\right|\left|\Psi(1)'\gamma\right|=0,$$

$$\left|\int_0^1 YY'\mathrm{d}u-\beta\int_0^1 Y(\mathrm{d}Y)'\delta\delta'\int_0^1(\mathrm{d}Y)Y'\right|=0. \tag{3.132}$$

By multiplying the left and right side of (3.132) with $|\delta'|$ and $|\delta|$ gives us,

$$\left|\delta\int_0^1 YY'\delta'\mathrm{d}u-\beta\delta\int_0^1 Y(\mathrm{d}Y)'\delta\delta'\int_0^1(\mathrm{d}Y)Y'\delta'\right|=0. \tag{3.133}$$

Now $G = \delta' Y$ and is a Brownian motion with variance $\delta' \zeta^{-1} \delta = I$. Therefore,

$$\left| \left[\int_0^1 GG' du \right] - \beta \left[\int_0^1 G(dG)' \right] \left[\int_0^1 (dG) G' \right] \right| = 0 . \qquad (3.134)$$

The result of Lemma 3.5 is found by noting that the solution of (3.127) is the reciprocal value of the solution to (3.134).

Q.E.D. (for Lemma 3.5)

Theorem 3.2: Under the hypothesis that there are h cointegrating vectors, the estimate of Θ and Ω_E are consistent, and the likelihood ratio test statistic of this hypothesis is asymptotically distributed as

$$tr \left\{ \left[\int_0^1 (dG) G' \right] \left\{ \int_0^1 GG' du \right\}^{-1} \left[\int_0^1 G(dG)' \right] \right\} \qquad (3.135)$$

where G is a $(k-h)$-dimensional Brownian motion with covariance matrix I.

Proof of Theorem 3.2:

From the expression for the likelihood ratio test statistic, the equation (3.47) can be expanded as

$$-2 \ln \Lambda_{agg} = -N \sum_{i=h+1}^{k} \ln\left(1 - \hat{\lambda}_i\right) = -N \sum_{i=h+1}^{k} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\left(-\hat{\lambda}_i\right)^j}{j} = N \sum_{i=h+1}^{k} \sum_{j=1}^{\infty} \frac{\left(\hat{\lambda}_i\right)^j}{j}$$

$$= N \sum_{i=h+1}^{k} \left[\hat{\lambda}_i + \sum_{j=2}^{\infty} \frac{\left(\hat{\lambda}_i\right)^j}{j} \right] = \sum_{i=h+1}^{k} N \hat{\lambda}_i + o_P(1)$$

because $\sum_{j=2}^{\infty} \frac{\left(\hat{\lambda}_i\right)^j}{j} \to 0$ as $j \to \infty$. Hence,

$$-2\ln\Lambda_{agg} = -N\sum_{i=h+1}^{k}\ln(1-\hat{\lambda}_i) = \sum_{i=h+1}^{k} N\hat{\lambda}_i + o_P(1). \tag{3.136}$$

Thus, from Lemma 3.5,

$$\begin{aligned} -2\ln\Lambda_{agg} &= -\mathrm{N}\sum_{i=h+1}^{k}\ln(1\text{-}\hat{\lambda}_\mathrm{i}) \\ &= \sum_{i=h+1}^{k}\mathrm{N}\hat{\lambda}_\mathrm{i} + \mathrm{o_P}(1) \xrightarrow{w} \mathrm{tr}\left\{\left[\int_0^1 (\mathrm{d}\boldsymbol{G})\boldsymbol{G}'\right]\left\{\int_0^1 \boldsymbol{G}\boldsymbol{G}'\mathrm{du}\right\}^{-1}\left[\int_0^1 \boldsymbol{G}(\mathrm{d}\boldsymbol{G})'\right]\right\}. \end{aligned} \tag{3.137}$$

The consistency of the estimator of $\boldsymbol{\Omega}_E$ is as follows:

The MLE of $\boldsymbol{\Omega}_E$ is given by

$$\begin{aligned} \hat{\boldsymbol{\Omega}}_E &= (\Im_{00} - \boldsymbol{F}_{00}) - [\Im_{0P} - \boldsymbol{F}_{0P}]\hat{\boldsymbol{A}}\left(\hat{\boldsymbol{A}}'[\Im_{PP} - \boldsymbol{F}_{PP}]\hat{\boldsymbol{A}}\right)^{-1}\hat{\boldsymbol{A}}'[\Im_{P0} - \boldsymbol{F}_{P0}] \\ &= (\Im_{00} - \boldsymbol{F}_{00}) - \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{A}}'[\Im_{P0} - \boldsymbol{F}_{P0}] \end{aligned}$$

By Lemma 3.3 a) $\Im_{00} \xrightarrow{a.s.} \Sigma_{00}$ and e) $\boldsymbol{A}'\Im_{PP}\boldsymbol{A} \xrightarrow{a.s.} \boldsymbol{A}'\Sigma_{PP}\boldsymbol{A}$. By equation (3.117), $\boldsymbol{F}_{00} \xrightarrow{a.s.} \sum_{i=1}^{Q}\Theta_i^2\boldsymbol{\Omega}_E$. Equations (3.114) and (3.115) give $\boldsymbol{A}'\boldsymbol{F}_{PP} \xrightarrow{a.s.} 0$ and $\boldsymbol{A}'\boldsymbol{F}_{P0} \xrightarrow{a.s.} 0$, so $\boldsymbol{F}_{0P}\boldsymbol{A} \xrightarrow{a.s.} 0$. Equations (3.123) and (3.124) state that $\hat{\boldsymbol{A}}' \xrightarrow{a.s.} \boldsymbol{A}'$ and $\hat{\boldsymbol{\alpha}} \xrightarrow{a.s.} \boldsymbol{\alpha} = \Sigma_{0P}\boldsymbol{A}(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})^{-1}$, so that

$$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{A}}(\Im_{PP} - \boldsymbol{F}_{PP})\hat{\boldsymbol{A}}'\hat{\boldsymbol{\alpha}} \xrightarrow{a.s.} \Sigma_{0P}\boldsymbol{A}(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})^{-1}(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})^{-1}\boldsymbol{A}'\Sigma_{P0} = \Sigma_{0P}\boldsymbol{A}(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})\boldsymbol{A}'\Sigma_{P0}$$

$$\begin{aligned} \hat{\boldsymbol{\Omega}}_E &= (\Im_{00} - \boldsymbol{F}_{00}) - [\Im_{0P} - \boldsymbol{F}_{0P}]\hat{\boldsymbol{A}}\left(\hat{\boldsymbol{A}}'[\Im_{PP} - \boldsymbol{F}_{PP}]\hat{\boldsymbol{A}}\right)^{-1}\hat{\boldsymbol{A}}'[\Im_{P0} - \boldsymbol{F}_{P0}] \\ &\xrightarrow{a.s.} \Sigma_{00} - \sum_{i=1}^{Q}\Theta_i^2\boldsymbol{\Omega}_E - \Sigma_{0P}\boldsymbol{A}(\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A})\boldsymbol{A}'\Sigma_{P0}. \end{aligned}$$

By Lemma 3.2, equation (3.76) $\Sigma_{00} = \boldsymbol{\alpha}\boldsymbol{A}'\Sigma_{PP}\boldsymbol{A}\boldsymbol{\alpha}' + \boldsymbol{\Omega}_E + \Theta_1^2\boldsymbol{\Omega}_E + \ldots + \Theta_Q^2\boldsymbol{\Omega}_E$, this means that $\Sigma_{00}-\sum_{i=1}^{Q}\Theta_i^2\Omega_E-\Sigma_{0P}A(A'\Sigma_{PP}A)A'\Sigma_{P0}=\Omega_E$. Hence,

$$\hat{\Omega}_E\xrightarrow{a.s.}\Omega_E.$$

The consistency of the estimators of Θ are then as follows:

The MLE of Θ is given as

$$\hat{\Theta}=\left(\hat{\alpha}\hat{A}'H_{PE}-H_{0E}\right)H_{EE}^{-1}$$

where $\hat{\alpha}=\left(\Im_{OP}\hat{A}-F_{0P}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}\right]^{-1}$.

$$\begin{aligned}\hat{\Theta}&=\left(\hat{\alpha}\hat{A}'H_{PE}-H_{0E}\right)H_{EE}^{-1}\\&=\left(\left(\Im_{0P}\hat{A}-F_{0P}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}\right]^{-1}\hat{A}'H_{PE}-H_{0E}\right)H_{EE}^{-1}\end{aligned}$$

By using Lemma 3.3 g) $A'H_{PE}\xrightarrow{a.s.}0$ and by equation (3.123) $\hat{A}\xrightarrow{a.s.}A$. This means that

$$\left(\Im_{0P}\hat{A}-F_{0P}\hat{A}\right)\left[\hat{A}'\left(\Im_{PP}-F_{PP}\right)\hat{A}\right]^{-1}\hat{A}'H_{PE}\xrightarrow{a.s.}0.$$

From Lemma 3.3 i) $H_{0E}\xrightarrow{a.s.}-\left(\Theta_1 I_k\ \cdots\ \Theta_Q I_k\right)\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}$ and h) $H_{EE}\xrightarrow{a.s.}\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}$. This means that $H_{EE}^{-1}\xrightarrow{a.s.}\left[\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}\right]^{-1}$. Therefore,

$$H_{0E}H_{EE}^{-1}\xrightarrow{a.s.}-\left(\Theta_1 I_k\ \cdots\ \Theta_Q I_k\right)\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}\left[\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}\right]^{-1}.$$

Then,

$$\hat{\Theta}\xrightarrow{a.s.}\left(0-\left[-\left(\Theta_1+\ldots+\Theta_Q\right)\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}\right]\right)\left(\operatorname{diag}\left(\Omega_E\right)_{Qk\times Qk}\right)^{-1}=\left(\Theta_1 I_k\ \cdots\ \Theta_Q I_k\right)=\Theta.$$

Q.E.D.

We showed that the asymptotic distribution given in equation (3.135) is the same limiting distribution given in Johansen (1988). This means that to test cointegration for aggregates we use the test statistic given in equation (3.47) with limiting distribution determined by Johansen (1988). Hence, we can use the same critical values that are determined by Johansen (1988).

If we recall the example given in Section 3, the inconsistent test results were obtained when we used an aggregate series to test for cointegration without considering the effects of aggregation. We now retest the cointegration for the aggregate series of Example 3.1 by using the adjusted test statistic; we get consistent conclusions as are summarized in Table 3.5. At all levels of aggregation, the test gives the same conclusion of cointegration in the system with rank 1.

3.5 An Empirical Example on Cointegration

To analyze the results of cointegration tests for basic and temporally aggregated series, U.S. wage and salary disbursements (WAGE) and U.S. real personal consumption expenditures (PCE) have been selected. The source of the data is U.S. Department of Commerce: Bureau of Economic Analysis. In the statistical analysis, we use SAS software and in order to obtain the results of modified test and the estimates that are developed in this chapter, we wrote a FORTRAN program with NAG FORTRAN library.

Table 3.5 Trace Test for Cointegration using the Adjusted Test Statistic When the Aggregate Series are Used

H_0: Rank =h	H_1: Rank > h	Eigenvalue	Trace	Critical Value
		$m = 1$		
0	0	0.5227	445.89	12.0
1	1	0.0036	2.16	4.2
		$m = 3$		
0	0	0.7969	318.81	12.0
1	1	0.0000	0.00	4.2
		$m = 4$		
0	0	0.8054	246.02	12.0
1	1	0.0035	0.53	4.2
		$m = 6$		
0	0	0.7713	147.98	12.0
1	1	0.0046	0.46	4.2
		$m = 8$		
0	0	0.7665	109.59	12.0
1	1	0.0067	0.50	4.2
		$m = 10$		
0	0	0.7231	77.06	12.0
1	1	0.0001	0.00	4.2
		$m = 12$		
0	0	0.6956	69.56	12.0
1	1	0.0063	0.32	4.2

3.5.1 Analysis of the Basic Series

The monthly U.S. wage and salary disbursements in billions of Dollars ($x_{1,t}$) and real personal consumption expenditures in billions of chained 2000 Dollars ($x_{2,t}$) are selected from January 1959 to December 2000, and there are 504 observations. Figure 3.1 shows the time series plot of the data. Both WAGE and PCE have an increasing trend and they are moving together.

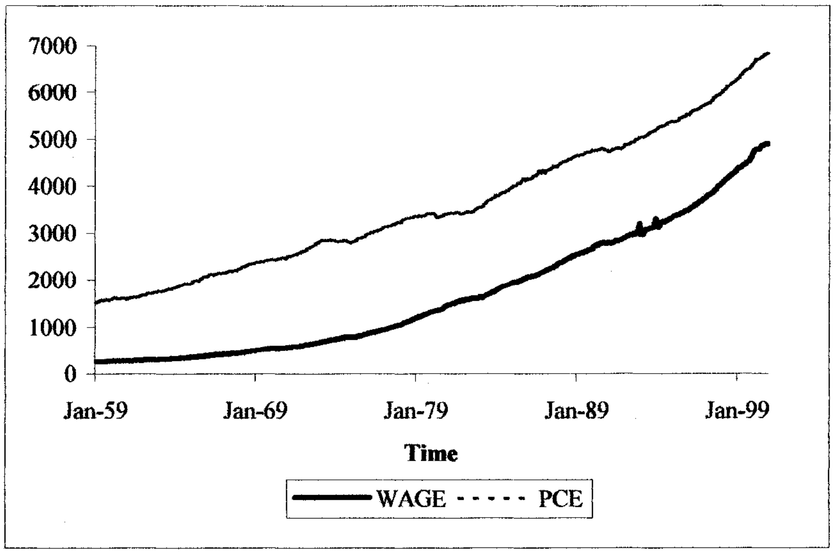


Figure 3.1 Time Series Plot of Monthly U.S. Wage-Salary Disbursements and Real Personal Consumption Expenditures

To be able to decide the order of the VARMA parameters, we looked at the minimum information criterion (MINIC) and partial cross correlations. MINIC is proposed by Quinn (1980), Spliid (1983) and Koreisha and Pukkila (1989) and is useful in identifying the orders of a VARMA (p, q) process. The partial cross-correlations have the cutoff property for a VAR(p) model, and so they can be useful in the identification of the order of a pure VAR structure (Ansley and Newbold ,1979). MINIC shown in Table 3.6 and the schematic representation of partial cross correlations given in Table 3.7 indicate that data are possibly generated either from a vector AR(2) or a vector AR(4) process. In the estimation of the VAR(4) model, estimates of the third and fourth AR parameter matrices are not significant so that we will consider vector AR(2) process.

Table 3.6 Minimum Information Criterion for Monthly WAGE and PCE

Lag	MA 0	MA 1	MA 2	MA 3	MA 4	MA 5
AR 0	29.361	29.394	29.407	29.420	29.433	29.445
AR 1	11.924	11.738	11.742	11.725	11.713	11.715
AR 2	11.781	11.735	11.727	11.725	11.722	11.722
AR 3	11.757	11.714	11.726	11.736	11.735	11.729
AR 4	11.699	11.717	11.730	11.739	11.736	11.720
AR 5	11.716	11.730	11.741	11.745	11.722	11.722

Table 3.7 Sample Partial Cross Correlations of Monthly WAGE and PCE

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
WAGE	+.	.-	..	..	..	..	..	..	..	..	..	..
PCE	.+	.+	..	..	..	..	..	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

To see whether x_{1t} and x_{2t} are stationary, the Dickey-Fuller unit root is conducted. Consider again, the AR(2) process:

$$x_{t,i} = \phi_1 x_{t-1,i} + \phi_2 x_{t-2,i} + a_{t,i}, \quad t = 1, \ldots, n, \tag{3.138}$$

with $x_{0,i} = 0$ and $\{a_{t,i}\} \sim$ i.i.d. $(0, \sigma_i^2)$, i = 1, 2. Equation (3.137) can be written as

$$x_{t,i} = \phi x_{t-1,i} + \delta \Delta x_{t-1,i} + a_{t,i}, \quad \text{i} = 1, 2. \tag{3.139}$$

Testing for a unit root in model (3.138) is equivalent to testing the null hypothesis in model (3.139) H_0 : $\phi = 1$ against the alternative hypothesis is H_1 : $|\phi| < 1$.

For this test, the test statistic is

$$n\left(\hat{\phi} - 1\right) / \left(1 - \hat{\delta}\right)^{-1}. \tag{3.140}$$

Dickey and Fuller also proposed an alternative test based on the OLS t-test of the null hypothesis H_0: $\phi = 1$,

$$\tau = \frac{\hat{\phi}-1}{s_{\hat{\rho}}} \xrightarrow{L} \frac{\frac{1}{2}\left\{[W(1)]^2 - 1\right\}}{\left\{\int [W(s)]^2\, ds\right\}^{1/2}}. \tag{3.141}$$

Table 3.8 gives the result of the Dickey-Fuller unit root test for τ (Tau). If we look at the p-values for both series, we can see that the probability is greater than 0.05. Therefore, this means that we can not reject H_0: $\phi = 1$; that is, the series is not stationary. And so, we can consider investigating whether there is cointegration in the system.

Table 3.8 The Dickey-Fuller Unit Root Test of Monthly WAGE and PCE

Variable	Type	Tau	Prob<Tau
WAGE	Zero Mean	15.10	0.9999
PCE	Zero Mean	14.17	0.9999

Cointegration is the phenomenon that each component $x_{i,t}$, $i = 1,\ldots,k$, of a vector time series process $\boldsymbol{x}_t$, is a unit root process, possibly with drift, but some linear combinations of the $x_{i,t}$'s is stationary. Thus

$$\boldsymbol{x}_t = \boldsymbol{\mu} + \boldsymbol{x}_{t-1} + \Gamma \Delta \boldsymbol{x}_{t-1} + \boldsymbol{a}_t \tag{3.142}$$

where $\boldsymbol{a}_t$ is a zero-mean k-variate stationary time series process and $\boldsymbol{\mu}$ is a k-vector of drift parameters (in our example, $\boldsymbol{\mu}$ is not significant, so we choose $\boldsymbol{\mu} = \boldsymbol{0}$), but there exists a $k\times h$ matrix $\boldsymbol{A}$ with rank $h<k$ such that $\boldsymbol{A}'\boldsymbol{x}_t$ is stationary.

In Table 3.9, H_0 is the null hypothesis and H_1 is the alternative hypothesis. The first row tests how $h = 0$ against $h > 0$; the second row tests how $h = 1$ against $h > 1$. The

Trace test statistics in the fourth column are computed by $-n\sum_{i=h+1}^{k}\log(1-\lambda_i)$ where n is the available number of observations and λ_i is the eigenvalue in the third column. The critical values at a 5% level of significance are used for testing. Hence, the above test results indicate that the series x_{1t} and x_{2t} are cointegrated with rank 1.

Table 3.9 The Trace Test for Cointegration of Monthly WAGE and PCE

H_0: Rank=h	H_1: Rank>h	Eigenvalue	Trace	Critical Value	Drift InECM	DriftIn Process
0	0	0.3882	250.76	12.0	NOINT	Constant
1	1	0.0082	4.11	4.2		

We can now write the error correction model ECM as

$$\Delta\boldsymbol{x}_t = \Gamma\Delta\boldsymbol{x}_{t-1} + \Pi\boldsymbol{x}_{t-2} + \boldsymbol{a}_t, \tag{3.143}$$

where $\boldsymbol{x}_t = (\mathrm{x}_{1t} \quad \mathrm{x}_{2t})'$, $\boldsymbol{a}_t = (\boldsymbol{a}_{1t} \quad \boldsymbol{a}_{2t})'$ and $\Pi = \gamma A'$.

The estimation of (3.143) leads to the following results:

$$\Delta\boldsymbol{x}_t = \begin{pmatrix} -0.31106 & 0.03675 \\ 0.00271 & -0.22775 \end{pmatrix}\Delta\boldsymbol{x}_{t-1} + \begin{pmatrix} 0.00282 & 0.00215 \\ 0.00278 & 0.00212 \end{pmatrix}\boldsymbol{x}_{t-2} + \boldsymbol{a}_t$$

with

$$\hat{A} = \begin{pmatrix} 1.000 \\ 0.764 \end{pmatrix}, \hat{\gamma} = \begin{pmatrix} 0.00282 \\ 0.00278 \end{pmatrix}, \text{ and } \hat{\Omega} = \begin{pmatrix} 347.424 & 29.679 \\ 29.67921 & 371.032 \end{pmatrix}.$$

3.5.2 Analysis of Temporally Aggregated Series

We now consider the quarterly U.S. wage and salary disbursements ($X_{1,T}$) and real personal consumption expenditures ($X_{2,T}$). Thus, the data are aggregated with $m = 3$. The sample size is now N = 126. Figure 3.2 presents the time series plot of the aggregated data. Both series have an increasing trend as in basic series. The data seem to be nonstationary.

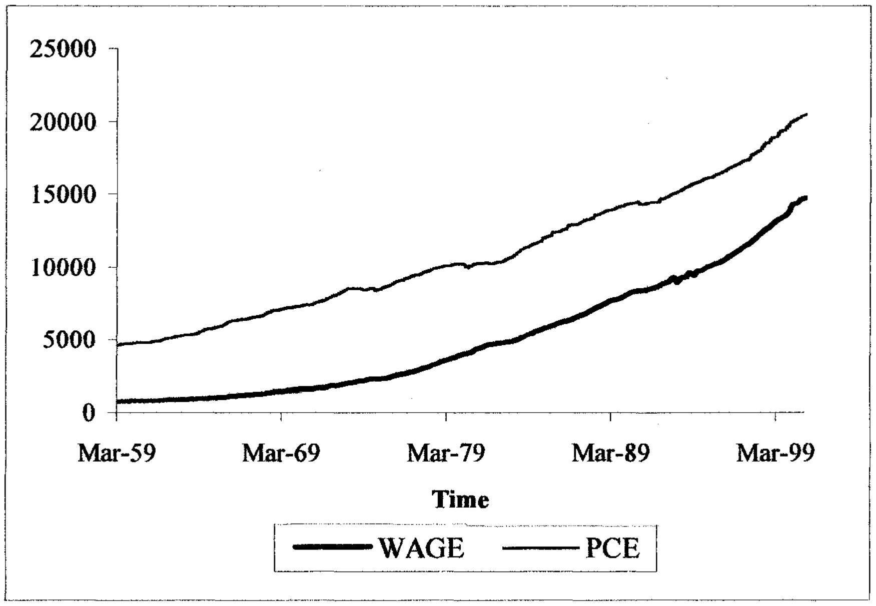


Figure 3.2 Time Series Plot of Quarterly U.S. Wage-Salary Disbursements and Real Personal Consumption Expenditures

To be able to decide the order of the VARMA parameters, we looked at the MINIC and partial cross correlations. The MINIC in Table 3.10 suggests a VAR(5) model while the schematic representation of partial cross correlations given in Table 3.11 suggests a vector AR(1) model. In light of the result from Chapter 2 that aggregation leads to a mixed ARMA model and VAR(2) is used for the basic series, we will consider a VARMA(2, 1) model for the aggregates.

Table 3.10 Minimum Information Criterion for Quarterly WAGE and PCE

Lag	MA 0	MA 1	MA 2	MA 3	MA 4	MA 5
AR 0	33.734	33.834	33.873	33.909	33.955	34.001
AR 1	16.802	16.829	16.805	16.728	16.687	16.678
AR 2	16.696	16.688	16.716	16.696	16.689	16.714
AR 3	16.662	16.706	16.711	16.714	16.722	16.718
AR 4	16.628	16.661	16.698	16.731	16.748	16.776
AR 5	16.604	16.649	16.701	16.734	16.769	16.769

Table 3.11 Sample Partial Cross Correlations of Quarterly WAGE and PCE

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
WAGE	+.	..	..	..	..	..	..	..	..	..	..	..
PCE	.+	..	..	..	..	..	..	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

We applied the unit root test to each series and the results are presented in Table 3.12. Since both p-values are greater than the significance level of 5%, the aggregated series is nonstationary at a 5% level of significance for $m = 3$.

Table 3.12 The Dickey-Fuller Unit Root Test of Quarterly WAGE and PCE

Variable	Type	Tau	Prob<Tau
WAGE	Zero Mean	13.45	0.9999
PCE	Zero Mean	8.36	0.9999

For the cointegration analysis, we first apply Johansen's Trace test to the data. The results given in Table 3.13 indicate that there is no cointegration in the system and the system is $I(1)$. This result contradicts with Theorem 3.1. This example also shows that we cannot use the unadjusted Johansen's trace test for aggregates.

Table 3.13 The Trace Test for Cointegration of Quarterly WAGE and PCE using Johansen's Test Statistic

H_0: Rank=h	H_1: Rank>h	Eigenvalue	Trace	Critical Value	Drift InECM	DriftIn Process
0	0	0.4638	111.19	12.0	NOINT	Constant
1	1	0.0454	7.72	4.2		

To be able to conduct a cointegration test for aggregates, we must first estimate the parameters of vector ARMA (2, 1) for aggregates. Since we need the error terms in order to calculate the test statistics, we fit the differenced series by maximum likelihood estimation method by using

$$\left(\boldsymbol{I}-\Phi_1\mathrm{B}-\Phi_2\mathrm{B}^2\right)\Delta\boldsymbol{X}_T=\left(\boldsymbol{I}-\Theta\mathrm{B}\right)\boldsymbol{E}_T.$$

The MA representation of this model can be found by

$$\Delta\boldsymbol{X}_T=\left(\boldsymbol{I}-\Phi_1\mathrm{B}-\Phi_2\mathrm{B}^2\right)^{-1}\left(\boldsymbol{I}-\Theta\mathrm{B}\right)\boldsymbol{E}_T.$$

The ECM is given by

$$\Delta\boldsymbol{X}_T=\eta\Delta\boldsymbol{X}_{T-1}+\Pi_{AG}\boldsymbol{X}_{T-2}+\boldsymbol{E}_T-\Theta\boldsymbol{E}_{T-1}, \qquad (3.134)$$

where $\boldsymbol{X}_T=\left(\mathrm{X}_{1T} \quad \mathrm{X}_{2T}\right)'$, $\boldsymbol{E}_T=\left(\mathrm{E}_{1T} \quad \mathrm{E}_{2T}\right)'$ and $\Pi_{AG}=\boldsymbol{\alpha}\boldsymbol{A}'$.

Since the error terms in VARMA model and ECM are the same, after maximum likelihood estimation of the parameters of VARMA model, we can obtain the residuals, $\hat{\boldsymbol{E}}_{T-1}$ and use them to calculate the test statistic given in equation (3.47). The results given in Table 3.14 imply that the aggregates are also cointegrated with rank 1 at 5% significance level.

Table 3.14 The Trace Test for Cointegration of Quarterly WAGE and PCE using Adjusted Test Statistic

H_0: Rank=h	H_1: Rank>h	Eigenvalue	Trace	Critical Value	Drift InECM	DriftIn Process
0	0	0.3059	61.82	12.0	NOINT	Constant
1	1	0.0028	0.47	4.2		

We can now write the error correction model as

$$\Delta \boldsymbol{X}_T = \eta \Delta \boldsymbol{X}_{T-1} + \Pi_{AG} \boldsymbol{X}_{T-2} + \boldsymbol{E}_t - \Theta \boldsymbol{E}_{T-1} \tag{3.144}$$

where $\boldsymbol{X}_T = \left(\mathrm{X}_{1T} \quad \mathrm{X}_{2T}\right)'$, $\boldsymbol{E}_T = \left(\mathrm{E}_{1T} \quad \mathrm{E}_{2T}\right)'$ and $\Pi_{AG} = \boldsymbol{\alpha} \boldsymbol{A}'$. The estimation of (3.144) gives the following error correction model for this data set:

$$\Delta \boldsymbol{X}_T = \begin{bmatrix} 1.077 & -0.187 \\ 1.198 & -0.164 \end{bmatrix} \Delta \boldsymbol{X}_{T-1} + \begin{bmatrix} 0.0097 & -0.0059 \\ -0.0046 & 0.0028 \end{bmatrix} \boldsymbol{X}_{T-2} + \boldsymbol{E}_t - \begin{bmatrix} 1.649 & -0.294 \\ 1.198 & -0.381 \end{bmatrix} \boldsymbol{E}_{T-1}$$

with

$$\hat{\boldsymbol{A}} = \begin{bmatrix} 4.813 \times 10^{-5} \\ 2.945 \times 10^{-5} \end{bmatrix}, \ \hat{\boldsymbol{\alpha}} = \begin{bmatrix} 201.2 \\ -95.7 \end{bmatrix}, \text{ and } \hat{\Omega}_E = \begin{bmatrix} 9397.5 & 22426.4 \\ 22426.4 & 130118.6 \end{bmatrix}.$$

CHAPTER 4

THE EFFECT OF TEMPORAL AGGREGATION ON GRANGER CAUSALITY

4.1. Introduction

The use of aggregate data for causal inference is common in the applied econometric literature. The most widely used causality tests are the Granger causality tests. There is considerable theoretical literature that investigates the influence of temporal aggregation on ARIMA models (Wei, 2006). A number of studies have also focused on temporal aggregation and dynamic relationships between variables and show that temporal aggregation weakens the distributed lag relationships (Telser 1967, Zellner and Montmarquette 1971, Sims 1971, Tiao and Wei 1976, Wei 1978a, Wei and Metha 1980). Tiao and Wei (1976) and Wei (1982) find that temporal aggregation turns one-way causality into a feedback system. Most of these studies consider the distributed lag models. Marcellino (1999) considers a vector model and shows that cointegration is invariant to temporal aggregation, but many other aspects such as seasonal unit roots, exogeneity, causality, impulse responses, trend-cycle components, measures of persistence and forecasting are affected by the aggregation process.

A time series $\{x_{1t}\}$ is said to cause another time series $\{x_{2t}\}$, if the present value of x_2 can be better predicted in terms of mean square error by using the past values of x_1 and x_2 rather than using only the past values of x_2.

Lütkepohl (1991) investigates the necessary and sufficient rules for non-causality between two groups of stationary time series variables. Consider the time series $\boldsymbol{x}_t$ with the MA representation

$$\boldsymbol{x}_t = \begin{bmatrix} \boldsymbol{x}_{1,t} \\ \boldsymbol{x}_{2,t} \end{bmatrix} = \begin{bmatrix} \Psi_{11}(\boldsymbol{B}) & \Psi_{12}(\boldsymbol{B}) \\ \Psi_{21}(\boldsymbol{B}) & \Psi_{22}(\boldsymbol{B}) \end{bmatrix} \boldsymbol{a}_t \tag{4.1}$$

where $\boldsymbol{x}_{i,t}$; $i=1,2$ are $k_i \times 1$, $i=1,2$ vector, $\boldsymbol{a}_t$ is a k-dimensional normal white noise vector with mean $\mathbf{0}$ and a covariance matrix Ω, and $\Psi_{ij}(B) = \sum_{\ell=0}^{\infty} \Psi_{ij,\ell} B^{\ell}$; $i,j=1,2$. Then, $\boldsymbol{x}_2$ does not cause $\boldsymbol{x}_1$ if and only if $\Psi_{12}(B) = \mathbf{0}$. Similarly, $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\Psi_{21}(B) = \mathbf{0}$.

4.2 Effects of Temporal Aggregation on the Causality Relationship between Two Sets of Variables

In the literature there are studies on the effects of temporal aggregation on the Granger causality. However, these studies do not take into consideration the fact that the form of the vector time series model changes after aggregation. For example, when the vector time series is generated from a VAR(1) process, aggregation changes the process into a VARMA(1,1) process as shown in Proposition 2.2. Thus, the non-causality conditions may not be the same for basic and aggregate series.

Consider the following two-dimensional VAR(1) process

$$(\boldsymbol{I} - \boldsymbol{\phi} B)\boldsymbol{x}_t = \boldsymbol{a}_t \tag{4.2}$$

$$\begin{bmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

where $\boldsymbol{a}_t$ vector is a sequence of i.i.d., random variables with mean vector $\mathbf{0}$ and covariance matrix Ω with MA representation

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{bmatrix}^{-1} \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = \frac{1}{(1-\phi_{11}B)(1-\phi_{22}B)-\phi_{12}\phi_{21}B^2} \begin{bmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{bmatrix} \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}.$$

By Lütkepohl (1991), the non-causality condition that x_1 does not cause x_2 for VAR(1) model is $\phi_{21} = 0$. As shown in Proposition 2.2, the corresponding aggregate model for the given basic model in (4.2) is the following two-dimensional VARMA(1,1) model

$$(\boldsymbol{I}-\varphi\mathrm{B})\boldsymbol{X}_T = (\boldsymbol{I}-\theta\mathrm{B})\boldsymbol{E}_T$$

$$\begin{bmatrix} 1-\varphi_{11}B & -\varphi_{12}B \\ -\varphi_{21}B & 1-\varphi_{22}B \end{bmatrix} \begin{bmatrix} X_{1T} \\ X_{2T} \end{bmatrix} = \begin{bmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1-\theta_{22}B \end{bmatrix} \begin{bmatrix} E_{1T} \\ E_{2T} \end{bmatrix} \tag{4.3}$$

where $\boldsymbol{E}_T$ vector is a sequence of i.i.d., random variables with mean vector $\mathbf{0}$ and covariance matrix Ω_E, $\varphi = \phi^m$ and θ is the solution of the following quadratic matrix equation

$$\theta^2\Gamma_1 + \theta\Gamma_0 + \Gamma_1' = \mathbf{0},$$

where

$$\Gamma_0 = \left[\sum_{j=0}^{m-1} \left(\sum_{i=0}^{j} \phi^i \right) \Omega \left(\sum_{i=0}^{j} \phi^i \right)' + \sum_{j=1}^{m-1} \phi^j \left(\sum_{i=0}^{m-1-j} \phi^i \right) \Omega \left(\sum_{i=0}^{m-1-j} \phi^i \right)' (\phi^j)' \right],$$

$$\Gamma_1 = \sum_{j=0}^{m-2} \left(\sum_{i=0}^{j} \phi^i \right) \Omega \left(\sum_{i=0}^{m-2-j} \phi^i \right)' (\phi^{j+1})',$$

and $\Omega_E = -\Gamma_1(\theta')^{-1}$. Hence, the MA representation of (4.3) is given by

$$\begin{bmatrix} X_{1T} \\ X_{2T} \end{bmatrix} = \begin{bmatrix} 1-\varphi_{11}B & -\varphi_{12}B \\ -\varphi_{21}B & 1-\varphi_{22}B \end{bmatrix}^{-1} \begin{bmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1-\theta_{22}B \end{bmatrix} \begin{bmatrix} E_{1T} \\ E_{2T} \end{bmatrix}$$

$$= \frac{1}{(1-\varphi_{11}B)(1-\varphi_{22}B)-\varphi_{12}\varphi_{21}B^2} \begin{bmatrix} 1-\varphi_{22}B & \varphi_{12}B \\ \varphi_{21}B & 1-\varphi_{11}B \end{bmatrix} \begin{bmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1-\theta_{22}B \end{bmatrix} \begin{bmatrix} E_{1T} \\ E_{2T} \end{bmatrix}$$

$$= \frac{1}{(1-\varphi_{11}B)(1-\varphi_{22}B)-\varphi_{12}\varphi_{21}B^2} \begin{bmatrix} (1-\varphi_{22}B)(1-\theta_{11}B)-\varphi_{12}\theta_{21}B^2 & -(1-\varphi_{22}B)\theta_{12}B+\varphi_{12}B(1-\theta_{22}B) \\ \varphi_{21}B(1-\theta_{11}B)-(1-\varphi_{11}B)\theta_{21}B & -\varphi_{21}\theta_{12}B^2+(1-\varphi_{11}B)(1-\theta_{22}B) \end{bmatrix} \begin{bmatrix} E_{1T} \\ E_{2T} \end{bmatrix}$$

This means that the non-causality condition X_1 does not cause X_2 for its aggregates are $\varphi_{21}B(1-\theta_{11}B)-(1-\varphi_{11}B)\theta_{21}B=0$, that is, $\varphi_{21}B-\varphi_{21}\theta_{11}B^2-\theta_{21}B-\varphi_{11}\theta_{21}B^2)=0$. So we can write the non-causality condition for aggregates as $\varphi_{21}-\theta_{21}=0$ and $\varphi_{11}\theta_{21}-\varphi_{21}\theta_{11}=0$, which means that $\varphi_{21}=\theta_{21}$ and $\varphi_{11}=\theta_{11}$.

The noncausality conditions for the basic model in (4.2) and for the aggregate model in (4.3) are clearly not the same. In general, the conditions on non-causality are different for basic series and aggregate series.

Example 4.1 To illustrate the test with basic series and aggregate series, let us consider the two-dimensional VAR(1) process in (4.2) with $\boldsymbol{\phi}=\begin{bmatrix} 0.6 & -0.5 \\ 0 & 0.8 \end{bmatrix}$ and $\boldsymbol{a}_t$ being normally distributed vector with zero mean and a covariance matrix $\Omega=\begin{bmatrix} 1.5 & 0.6 \\ 0.6 & 2.0 \end{bmatrix}$. The aggregate model for m = 2 is obtained as VARMA(1,1) with $\boldsymbol{\varphi}=\boldsymbol{\phi}^2=\begin{bmatrix} 0.36 & -0.7 \\ 0 & 0.64 \end{bmatrix}$, $\boldsymbol{\theta}=\begin{bmatrix} -0.11 & 0.07 \\ -0.09 & -0.17 \end{bmatrix}$ and $\Omega_E=\begin{bmatrix} 5.45 & 0.07 \\ 0.07 & 9.42 \end{bmatrix}$. It can be seen that the non-causality condition that X_1 does not cause X_2 is not satisfied because $\varphi_{21}=0\neq\theta_{21}=-0.09$ and $\varphi_{11}=0.36\neq\theta_{11}=-0.11$. To see the consequence of using aggregates in testing causality, we obtain a simulation of a basic series of 15,120 observations from the given VAR(1) process in (4.2) with the above parameter matrices. The series are then aggregated for various *m*, and likelihood ratio test introduced in Section 1.5.1.1 is performed for each aggregation period of *m* from 1 to 12. The p-values of these tests are summarized in Table 4.1.

Table 4.1 The p-values of the Likelihood Ratio Test for Non-causality Using Aggregate Series

M	p-value	*m*	p-value
1	0.234922	7	7.4E-214
2	0	8	1.3E-170
3	0	9	3.2E-135
4	0	10	9.94E-97
5	0	11	1.92E-84
6	3.2E-308	12	2.96E-65

Table 4.1 shows that for the basic series, when $m = 1$, as expected, there is no causal relationship from x_1 to x_2 at a 5% level of significance. However, when aggregate series are used, the test indicates a causal relationship from X_1 to X_2 at all levels of aggregation. The similar result was derived earlier by Tiao and Wei (1976), and Wei (1982) in terms of distributed lag models.

4.3 Temporal Aggregation and Granger Non-Causality Tests in Cointegrated Systems

In a vector autoregressive process, the Granger non-causality of one set of variables for another set is characterized by the number of constraints on the autoregressive coefficients. If the process is stationary, the test for non-causality is usually performed using either Wald test or the likelihood ratio test which are asymptotically distributed as chi-square distributions. Mosconi and Giannini (1992) suggest a likelihood ratio test which is more efficient in order to test non-causality in cointegrated systems because it imposes the cointegration constraints upon both the null and the alternative hypothesis. Therefore, we will use their approach to test causality of aggregate series in cointegrated systems.

In Chapter 3, we have showed that the well-known cointegration test, trace test, fails to detect cointegration for aggregates because it does not consider the model change due to aggregation. The test for non-causality that Mosconi and Giannini developed also fails to consider this effect. Although their likelihood ratio test for non-causality is efficient by imposing the cointegration constraints upon both the null and the alternative hypotheses, it is not suitable for testing non-causality in cointegrated system using aggregate series. In this section, we will develop a modified test statistic to test non-causality in cointegrated system for the aggregate series.

Given a k-dimensional cointegrated $I(1)$ series $\boldsymbol{x}_t$ that follows a vector autoregressive VAR(p) process:

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t. \tag{4.4}$$

where $\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I} - \boldsymbol{\phi}_1\mathrm{B} - \ldots - \boldsymbol{\phi}_p\mathrm{B}^p$ and $\boldsymbol{\phi}_i$'s are $k \times k$ matrices, $\boldsymbol{a}_t$'s are i.i.d. k-dimensional random vectors with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Omega}$. We have shown in (3.11) that $(1-\mathrm{B})\boldsymbol{X}_T$ is $I(0)$, and its Wold representation is given by

$$(1-\mathrm{B})\boldsymbol{X}_T = \Psi(\mathrm{B})\boldsymbol{E}_T = \sum_{i=0}^{\infty}\Psi_i\boldsymbol{E}_{T-i}, \tag{4.5}$$

where $\sum_i |\Psi_i| < \infty$ and $\Psi(1) \neq \mathbf{0}$, and can be obtained through the relationship

$$\Psi(\mathrm{B})\boldsymbol{E}_T=\left(1+\mathrm{B}+\cdots+\mathrm{B}^{m-1}\right)^2\left[\frac{\phi_p(\mathrm{B})}{(1-\mathrm{B})}\right]^{-1}\boldsymbol{a}_{mT}\,. \tag{4.6}$$

Furthermore, as shown in (3.13), we have

$$\boldsymbol{\Phi}_P(\mathrm{B})\boldsymbol{X}_T=\Theta_\mathrm{Q}(\mathrm{B})\boldsymbol{E}_T \tag{4.7}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_P(\mathrm{B})&=\mathrm{adj}(\Psi(\mathrm{B}))/(1-\mathrm{B})^{h-1}\\&=\boldsymbol{I}-\boldsymbol{\Phi}_1\mathrm{B}-\ldots-\boldsymbol{\Phi}_P\mathrm{B}^\mathrm{P}\end{aligned}$$

and $$\Theta_\mathrm{Q}(\mathrm{B})=\det\left(\Psi(\mathrm{B})\right)/(1-\mathrm{B})^h=1\text{-}\Theta_1 B-\ldots-\Theta_Q B^Q\,.$$

By Granger representation theorem, the error correction representation of the basic series is given by

$$\Delta\boldsymbol{x}_t=\sum_{i=1}^{p-1}\Gamma_i\Delta\boldsymbol{x}_{t-i}+\Pi\boldsymbol{x}_{t-p}+\boldsymbol{a}_t \tag{4.8}$$

whereas the error correction representation of the aggregates as shown in (3.14) is given by

$$\Delta\boldsymbol{X}_T=\sum_{i=1}^{P-1}\boldsymbol{\eta}_i\Delta\boldsymbol{X}_{T-i}+\Pi_{AG}\boldsymbol{X}_{T-P}+\Theta_\mathrm{Q}(\mathrm{B})\boldsymbol{E}_T\,, \tag{4.9}$$

where $\boldsymbol{\eta}_i=-\boldsymbol{I}+\boldsymbol{\Phi}_1+\cdots+\boldsymbol{\Phi}_i$, $i=1,\ldots,P-1$, $\Pi_{AG}=-\boldsymbol{\Phi}_P(1)=\boldsymbol{\alpha}\boldsymbol{A}'$ for some $\boldsymbol{\alpha}$.

The test statistic and its distribution derived from the error correction presentation given in Equation (4.9) are summarized in the following theorem.

Theorem 4.1 For the basic generating process given in (4.4), let $\boldsymbol{X}_T$ be the corresponding aggregate vector series and $\boldsymbol{X}_T=(\underset{1\times k_1}{\boldsymbol{X}'_{T,1}},\underset{1\times k_2}{\boldsymbol{X}'_{T,2}})'$ where $k=k_1+k_2$. To test the null hypothesis that $\boldsymbol{X}_1$ does not cause $\boldsymbol{X}_2$ is equivalent to test

$$H_0(h,h_1,h_2)\text{: } U'\eta V = 0, \quad U'\Pi_{AG}U_\perp = 0, \quad \Pi_{AG} = \alpha A'$$

where $U = \left[0'_{k\text{-}k_2\times k_2} \quad I'_{k_2}\right]'_{k\times k_2}$, $\eta = \left[\eta'_1 \quad \cdots \quad \eta'_{P-1}\right]'$, $V_{k(P-1)\times k_1(P-1)} = I_{P-1} \otimes U_\perp$,

$\underset{k\times k}{\Pi_{AG}} = \alpha A'$ and $U_\perp = \left[I'_{k_1} \quad 0'_{k_1\times k\text{-}k_1}\right]'_{k\times k_1}$, against the alternative hypothesis,

$$H_A(h)\text{: } \Pi_{AG} = \alpha A'$$

where α is the full rank $k \times h$ adjustment matrix and A is the full rank $k \times h$ cointegrating matrix. The likelihood ratio non-causality test statistic is given by

$$-2\ln \frac{\max_{H_0(h,h_1,h_2)} L\left[\eta, \Pi_{AG}, \Theta; X_1, \cdots, X_N\right]}{\max_{H_A(h)} L\left[\eta, \Pi_{AG}, \Theta; X_1, \cdots, X_N\right]}, \tag{4.10}$$

and it follows an asymptotical χ^2 distribution with $kh - k_1h_1 - k_2h_2 - h_1h_2 + k_1k_2(P-1)$ degrees of freedom.

Proof for Theorem 4.1:

Let us rewrite Equation (4.9) more explicitly as follows:

$$\Delta X_T = \sum_{i=1}^{P-1} \eta_i \Delta X_{T-i} + \Pi_{AG} X_{T-P} + E_T - \Theta_1 E_{T-1} - \cdots - \Theta_Q E_{T-Q}, \tag{4.11}$$

where $\eta_i = -I + \Phi_1 + \cdots + \Phi_i$, $i = 1, \ldots, P-1$, $\Pi_{AG} = -\Phi_P(1) = \alpha A'$ for some α.

First, let us partition X_T as $X_T = (\underset{1\times k_1}{X'_{T,1}}, \underset{1\times k_2}{X'_{T,2}})'$ where $k = k_1 + k_2$. Then, the MA representation of the partition can be written as

$$(1-B)X_T = \begin{bmatrix} \Psi_{11}(B) & \Psi_{12}(B) \\ \Psi_{21}(B) & \Psi_{22}(B) \end{bmatrix} E_T,$$

and Equation (4.11) becomes

$$\Delta X_T = \sum_{i=1}^{P-1} \begin{bmatrix} \eta_{11.i} & \eta_{12.i} \\ \eta_{21.i} & \eta_{22.i} \end{bmatrix} \Delta X_{T-i} + \begin{bmatrix} \Pi_{11.AG} & \Pi_{12.AG} \\ \Pi_{21.AG} & \Pi_{22.AG} \end{bmatrix} X_{T-P} + \Theta_Q(B) E_T . \tag{4.12}$$

By the results given in section 1.5, X_1 does not cause X_2 if $\Psi_{21}(B) = \boldsymbol{0}$. This means that $\eta_{21,i} = \boldsymbol{0}$ and $\Pi_{21,AG} = \boldsymbol{0}$ in equation (4.12). In this framework, X_1 does not cause X_2 if the hypothesis

$$H_0 \text{: } U'\eta V = \boldsymbol{0}, \quad U'\Pi_{AG}U_\perp = \boldsymbol{0} \tag{4.13}$$

holds where $U = \begin{bmatrix} \boldsymbol{0}'_{k-k_2 \times k_2} & I'_{k_2} \end{bmatrix}'_{k \times k_2}$, $\eta = \begin{bmatrix} \eta'_1 & \cdots & \eta'_{P-1} \end{bmatrix}'$, $V_{k(P-1) \times k_1(P-1)} = I_{P-1} \otimes U_\perp$, $\underset{k \times k}{\Pi_{AG}} = \alpha A'$ and $U_\perp = \begin{bmatrix} I'_{k_1} & 0'_{k_1 \times k-k_1} \end{bmatrix}'_{k \times k_1}$.

Following the results in Mosconi and Giannini (1992), we can show that for a given reduced rank matrix $\underset{k \times k}{\Pi_{AG}} = \alpha A'$, $U'\Pi_{AG}U_\perp$ is equal to zero matrix if and only if

$$\alpha = \begin{bmatrix} U_\perp \alpha_{11} & | \alpha_2 \end{bmatrix} \text{ and } A = \begin{bmatrix} A_1 & | U A_{22} \end{bmatrix} \tag{4.14}$$

where α_{11} is $k_1 \times h_1$, α_2 is $k \times h_2$, A_1 is $k \times h_1$ and A_{22} is $k_2 \times h_2$ with $h = h_1 + h_2$.

Let's partition $\alpha_2 = \begin{bmatrix} \alpha'_{12} & \alpha'_{22} \end{bmatrix}'$ and $A_1 = \begin{bmatrix} A'_{11} & A'_{21} \end{bmatrix}'$ where α_{12} is $k_1 \times h_2$, α_{22} is $k_2 \times h_2$, A_{11} is $k_1 \times h_1$ and A_{21} is $k_2 \times h_1$. This means that

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Therefore,

$$\Pi_{AG} = \alpha A' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} A'_{11} & A'_{21} \\ 0 & A'_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} A'_{11} & \alpha_{11} A'_{21} + \alpha_{12} A'_{22} \\ 0 & \alpha_{22} A'_{22} \end{bmatrix}.$$

This implies that h_1 is rank($\Pi_{AG,11}$) and h_2 is rank($\Pi_{AG,22}$), and together they determine the distribution of unit roots for the whole system.

The null hypothesis of non-causality in the aggregate cointegrated systems $\mathrm{H_0(h,h_1,h_2)}$ defined by Mosconi and Giannini (1992) becomes

$$\mathrm{H_0(h,h_1,h_2)}: \boldsymbol{U'\eta V = 0}, \quad \boldsymbol{U'}\Pi_{AG}\boldsymbol{U_\perp = 0}, \quad \Pi_{AG} = \boldsymbol{\alpha A'},$$

and the alternative hypothesis is $\mathrm{H_A(h)}: \Pi_{AG} = \boldsymbol{\alpha A'}$ where $\boldsymbol{\alpha}$ is the full rank $k \times h$ adjustment matrix, and $\boldsymbol{A}$ is the full rank $k \times h$ cointegrating matrix. By applying the cointegration restrictions to both the null and the alternative hypotheses, a more efficient test can be obtained.

Under the null hypothesis $\mathrm{H_0(h,h_1,h_2)}$, Equation (4.9) can be rewritten as

$$\Delta \boldsymbol{X_T} = \sum_{i=1}^{P-1} \eta_i \Delta \boldsymbol{X_{T-i}} + (\boldsymbol{U_\perp \alpha_{11} A_1' + \alpha_2 A_{22}' U'})\boldsymbol{X_{T-P}} + \boldsymbol{E_T} - \Theta(\mathrm{B})\boldsymbol{E}. \tag{4.15}$$

Let $\boldsymbol{Z_{0T}} = [\Delta \boldsymbol{X_T}]_{k\times 1}$, $\boldsymbol{Z_{1T}} = (\Delta \boldsymbol{X'_{T-1}}, \dots, \Delta \boldsymbol{X'_{T-P+1}})'_{k(P-1)\times 1}$, $\boldsymbol{Z_{PT}} = [\boldsymbol{X_{T-P}}]_{k\times 1}$, $\eta = (\eta_1, \dots, \eta_{P-1})_{k\times k(P-1)}$, $\Theta = (\Theta_1 I_k \ \cdots \ \Theta_Q I_k)_{k\times Qk}$ and $\boldsymbol{E'} = (\mathrm{E'_{T-1}} \ \cdots \ \mathrm{E'_{T-Q}})'_{Qk\times 1}$.

Then, Equation (4.15) becomes

$$\boldsymbol{Z_{0T}} = \eta \boldsymbol{Z_{1T}} + (\boldsymbol{U_\perp \alpha_{11} A_1' + \alpha_2 A_{22}' U'})\boldsymbol{Z_{PT}} + \boldsymbol{E_T} - \Theta \boldsymbol{E}, \tag{4.16}$$

where $\boldsymbol{U'\eta V = 0}$. Let also $\mathrm{M}_{ij} = \mathrm{N}^{-1}\sum_{T=1}^{N} \boldsymbol{Z_{iT} Z'_{jT}}$, (i,j=0,1,P), $\aleph_{iE} = \mathrm{N}^{-1}\sum_{T=1}^{N} \boldsymbol{Z_{iT} E'}$, (i=0,1,P) and $\aleph_{EE} = \mathrm{N}^{-1}\boldsymbol{EE'}$, $\Im_{ij} = \mathrm{M}_{ij} - \mathrm{M}_{i1}\mathrm{M}_{11}^{-1}\mathrm{M}_{1j}$, (i,j=0,P), $\boldsymbol{H_{iE}} = \aleph_{iE} - \mathrm{M}_{i1}\mathrm{M}_{11}^{-1}\aleph_{1E}$, (i = 0, P), $\boldsymbol{H_{EE}} = \aleph_{EE} - \aleph_{E1}\mathrm{M}_{11}^{-1}\aleph_{1E}$.

Since the error terms $\boldsymbol{E_T}$'s follow a multivariate normal, the likelihood function is proportional to

$$|\Omega_E|^{-N/2}\exp\left\{-\frac{1}{2}\left(Z_{0T}-\eta Z_{1T}-(U_\perp\alpha_{11}A_1'+\alpha_2A_{22}'U')Z_{PT}+\Theta E\right)'\Omega_E^{-1}\times\right. \\ \left.\times\left(Z_{0T}-\eta Z_{1T}-(U_\perp\alpha_{11}A_1'+\alpha_2A_{22}'U')Z_{PT}+\Theta E\right)/2\right\}. \quad (4.17)$$

The estimation of the parameters does not easily reduce to an eigenvalue problem which is discussed in Chapter 3. Here, we use an iterative algorithm to maximize the likelihood function. The algorithm consists of the following steps:

1. Estimate the parameters α_2 and A_{22} by setting $\hat{\alpha}_{11}=\mathbf{0}$, $\hat{A}_1=\mathbf{0}$, $\hat{\eta}=\mathbf{0}$, and $\hat{\Theta}=\mathbf{0}$.
2. For the given values of $\alpha_2=\hat{\alpha}_2$ and $A_{22}=\hat{A}_{22}$, estimate α_{11}, A_1, η, and Θ.
3. For the given values of $\alpha_{11}=\hat{\alpha}_{11}$, $A_1=\hat{A}_1$, $\Theta=\hat{\Theta}$, and $\eta=\hat{\eta}$, estimate α_2 and A_{22}.
4. Continue with steps 2 and 3 until convergence.

Let $\hat{\alpha}_{11,i},\hat{A}_{1,i},\hat{\alpha}_{2,i},\hat{A}_{22,i},\hat{\eta}_i,\hat{\Theta}_i$ and $\hat{\Omega}_{E,i}$ be the i-th step estimates of the corresponding parameters. The iteration is initialized by setting $\hat{\alpha}_{11,1}=\mathbf{0},\hat{A}_{1,1}=\mathbf{0}$, $\hat{\eta}_1=\mathbf{0}$ and $\hat{\Theta}_1=\mathbf{0}$. Then Equation (4.16) becomes

$$Z_{0T}=\alpha_2A_{22}'U'Z_{PT}+E_T.$$

The log-likelihood function of errors is given by

$$L\left(\alpha_{2,i},A_{22,i},\Omega_{E,i}\right)\propto -N/2\ln|\Omega_E|-\left\{\sum_{T=1}^{N}\mathrm{tr}\left[\Omega_E^{-1}\left(Z_{0T}-\alpha_2A_{22}'U'Z_{PT}\right)\left(Z_{0T}-\alpha_2A_{22}'U'Z_{PT}\right)'\right]/2\right\}.$$

For a fixed value A_{22}, $\hat{\alpha}_2(A_{22})$ and $\hat{\Omega}_E(A_{22})$ are calculated as

$$\hat{\alpha}_{2.1}\left(A_{22}\right)=\Im_{0P}UA_{22}\left(A_{22}'U'\Im_{PP}UA_{22}\right)^{-1} \quad (4.18)$$

and

$$\hat{\Omega}_{E.1}(A_{22}) = \Im_{00} - \Im_{0P}UA_{22}\left(A'_{22}U'\Im_{PP}UA_{22}\right)^{-1}A'_{22}U'\Im_{0P}. \tag{4.19}$$

By following the same steps in Section 3.4.1, the maximum likelihood can be obtained by choosing $\hat{A}_{22,i}$ to be the first h_2 eigenvectors of $\left(U'\Im_{0P}\Im_{00}^{-1}\Im_{0P}U\right)\left(U'\Im_{PP}U\right)^{-1}$. Then the MLEs of $\alpha_{2.1}$ and $\Omega_{E.1}$ becomes

$$\hat{\alpha}_{2.1} = \Im_{0P}U\hat{A}_{22,i} \tag{4.20}$$

and

$$\hat{\Omega}_{E.1} = \Im_{00} - \Im_{0P}U\hat{A}_{22,i}\hat{A}'_{22,i}U'\Im_{0P}. \tag{4.21}$$

The maximized likelihood is proportional to $\left|\hat{\Omega}_{E,i}\right|^{-N/2}$ and is given by

$$\mathrm{L}_{\max,i}^{-2/N} = \left|\Im_{00}\right|\left|I - \hat{A}'_{22,i}U'\Im_{0P}\Im_{00}^{-1}\Im_{0P}U\hat{A}_{22,i}\right|. \tag{4.22}$$

By using the relationship between eigenvalues and eigenvectors, we obtain

$$U'\Im_{0P}\Im_{00}^{-1}\Im_{0P}U\hat{A}_{22,i} = U'\Im_{PP}U\hat{A}_{22,i}\mathrm{T}_{A_{22,i}}$$

where $\mathrm{T}_{A_{22},i}$ denotes the diagonal matrix of ordered eigenvalues and $\hat{A}_{22,i}$ is the matrix of the corresponding eigenvectors. Hence, Equation (4.22) becomes

$$\mathrm{L}_{\max,i}^{-2/N} = \left|\Im_{00}\right|\left|I_{h_2} - \hat{A}_{22,i}{}'U'\Im_{PP}U\hat{A}_{22,i}\mathrm{T}_{A_{22,i}}\right|$$

and since $\hat{A}'_{22,i}U'\Im_{PP}U\hat{A}_{22,i} = I_{h_2}$, we have

$$\mathrm{L}_{\max,i}^{-2/N} = \left|\Im_{00}\right|\left|I_{h_2} - \mathrm{T}_{A_{22,i}}\right|.$$

Therefore, the maximized likelihood function is given by

$$\mathrm{L}_{\max,i}^{-2/N} = \left|\Im_{00}\right|\prod_{j=1}^{h_2}(1-\hat{\lambda}_{j,i}) \tag{4.23}$$

where $\hat{\lambda}_{j,i}$'s are ordered eigenvalues $1 > \hat{\lambda}_{1,i} > \ldots > \hat{\lambda}_{k,i} > 0$ of $\left(U'\Im_{0P}\Im_{00}^{-1}\Im_{0P}U\right)\left(U'\Im_{PP}U\right)^{-1}$.

The next step of the estimation is to find the MLEs of $\alpha_{11,i+1}, A_{1,i+1}$ and $\hat{\Theta}_{i+1}$ by using the MLEs of $\alpha_{2,i}$ and $A_{22,i}$ from the i-th step. Equation (4.16) now becomes

$$Z_{0T} - \hat{\Pi}_2 Z_{PT} = \eta Z_{1T} + U_{\perp}\alpha_{11}A_1' S_{PT} + E_T - \Theta E \tag{4.24}$$

where $\hat{\Pi}_2 = \hat{\alpha}_{2.i}\hat{A}_{22.i}' U'$.

For fixed values of α_{11}, A_1, and Θ, the maximum likelihood estimation consists of a regression of $Z_{0T} - (U_{\perp}\alpha_{11}A_1' + \hat{\Pi}_2 U')Z_{PT} + \Theta E$ on Z_{1T} giving equations

$$\sum_{T=1}^{N} Z_{0T}Z_{1T}' = \eta\sum_{T=1}^{N} Z_{1T}Z_{1T}' + (U_{\perp}\alpha_{11}A_1' + \hat{\Pi}_2 U')\sum_{T=1}^{N} Z_{PT}Z_{1T}' - \Theta\sum_{T=1}^{N} EZ_{1T}' \tag{4.25}$$

where $\sum_{T=1}^{N} E_T Z_{1T}' = \mathbf{0}$. By using the product moment, matrices Equation (4.25) can be written as

$$\mathrm{M}_{01} = \eta \mathrm{M}_{11} + (U_{\perp}\alpha_{11}A_1' + \hat{\Pi}_2 U')\mathrm{M}_{P1} - \Theta\aleph_{E1} \tag{4.26}$$

or

$$\eta = \mathrm{M}_{01}\mathrm{M}_{11}^{-1} - (U_{\perp}\alpha_{11}A_1' + \alpha_2 A_{22}' U')\mathrm{M}_{P1}\mathrm{M}_{11}^{-1} + \Theta\aleph_{E1}\mathrm{M}_{11}^{-1}\ . \tag{4.27}$$

Let's define

$$R_{0T} = Z_{0T} - \mathrm{M}_{01}\mathrm{M}_{11}^{-1}Z_{1T}, \tag{4.28}$$

$$R_{PT} = Z_{PT} - \mathrm{M}_{P1}\mathrm{M}_{11}^{-1}Z_{1T}, \tag{4.29}$$

$$R_E = E - \aleph_{E1}\mathrm{M}_{11}^{-1}Z_{1T}, \tag{4.30}$$

and also define $\Im_{ij} = \boldsymbol{R}_{ij}\boldsymbol{R}'_{ij}$, $\boldsymbol{H}_{iE} = \boldsymbol{R}_{ij}\boldsymbol{R}'_{E}$ (i, j = 0, P), and $\boldsymbol{H}_{EE} = \boldsymbol{R}_{E}\boldsymbol{R}'_{E}$. Then Equation (4.25) becomes

$$\boldsymbol{R}_{0T} = (\boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1} + \hat{\Pi}_{2}\boldsymbol{U}')\boldsymbol{R}_{PT} + \boldsymbol{E}_{T} - \boldsymbol{\Theta}\boldsymbol{R}_{E}. \tag{4.31}$$

The log-likelihood function of $\boldsymbol{E}_T$ condition on $\hat{\Pi}_2\boldsymbol{R}_{PT}$ is given by

$$\begin{aligned} &\ln \boldsymbol{L}(\boldsymbol{\alpha}_{11}, \boldsymbol{A}_1, \boldsymbol{\Theta}, \Omega_{i+1}) \propto \\ &-\frac{N}{2}\ln|\Omega_{i+1}| - \{\boldsymbol{tr}\left[\Omega_{i+1}^{-1}\left(\boldsymbol{R}_{0T} - \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\boldsymbol{R}_{PT} - \hat{\Pi}_{2}\boldsymbol{R}_{PT} + \boldsymbol{\Theta}\boldsymbol{R}_{E} - \boldsymbol{W}\hat{\Pi}_{2}\boldsymbol{R}_{PT}\right)\times\right. \\ &\left.\left.\times\left(\boldsymbol{R}_{0T} - \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\boldsymbol{R}_{PT} - \hat{\Pi}_{2}\boldsymbol{R}_{PT} + \boldsymbol{\Theta}\boldsymbol{R}_{E} - \boldsymbol{W}\hat{\Pi}_{2}\boldsymbol{R}_{PT}\right)'/2\right]\right\} \end{aligned} \tag{4.32}$$

where $\boldsymbol{W} = \Omega_{E,i+1,i}\Omega_{E,i}^{-1}$. The estimator of $\boldsymbol{W}$ is given by

$$\begin{aligned} \frac{\partial \ln \boldsymbol{L}(\boldsymbol{\alpha}_{11}, \boldsymbol{A}_1, \boldsymbol{\Theta}, \Omega_{i+1})}{\partial \boldsymbol{W}} &= -\Im_{0P}\hat{\Pi}'_{2} + \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\Im_{PP}\hat{\Pi}'_{2} + \hat{\Pi}_{2}\Im_{PP}\hat{\Pi}'_{2} \\ &-\boldsymbol{\Theta}\boldsymbol{H}_{EP}\hat{\Pi}'_{2} + \boldsymbol{W}\left(\hat{\Pi}_{2}\Im_{PP}\hat{\Pi}'_{2}\right) = \boldsymbol{0} \end{aligned}.$$

$$\hat{\boldsymbol{W}} = \left(\Im_{0P}\hat{\Pi}'_{2} - \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\Im_{PP}\hat{\Pi}'_{2} - \hat{\Pi}_{2}\Im_{PP}\hat{\Pi}'_{2} + \boldsymbol{\Theta}\boldsymbol{H}_{EP}\hat{\Pi}'_{2}\right)\left(\hat{\Pi}_{2}\Im_{PP}\hat{\Pi}'_{2}\right)^{-1}. \tag{4.33}$$

Replacing $\boldsymbol{W}$ by Equation (4.33) in the following function that is part of the log-likelihood function given in Equation (4.32)

$$\boldsymbol{R}_{0T} - \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\boldsymbol{R}_{PT} - \hat{\Pi}_{2}\boldsymbol{R}_{PT} + \boldsymbol{\Theta}\boldsymbol{R}_{E} - \boldsymbol{W}\hat{\Pi}_{2}\boldsymbol{R}_{PT}$$

gives

$$\boldsymbol{R}_{0T} - \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\boldsymbol{R}_{PT} + \boldsymbol{\Theta}\boldsymbol{R}_{E} - \Im_{0P}\Im_{PP.\hat{\Pi}_2}\boldsymbol{R}_{PT} + \boldsymbol{U}_{\perp}\boldsymbol{\alpha}_{11}\boldsymbol{A}'_{1}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\boldsymbol{R}_{PT} - \boldsymbol{\Theta}\boldsymbol{H}_{EP}\Im_{PP.\hat{\Pi}_2}\boldsymbol{R}_{PT} \tag{4.34}$$

where $\Im_{PP.\hat{\Pi}_2} = \hat{\Pi}'_{2}\left(\hat{\Pi}_{2}\Im_{PP}\hat{\Pi}'_{2}\right)^{-1}\hat{\Pi}_{2}$.

By following Johansen and Juselius (1990), Equation (4.34) can be expressed in variables as

$$U_\perp' R_{0T} - U_\perp' U_\perp \alpha_{11} A_1' R_{PT} + U_\perp' \Theta R_E - U_\perp' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} + \\ + U_\perp' U_\perp \alpha_{11} A_1' \Im_{PP} \Im_{PP.\hat{\Pi}_2} R_{PT} - U_\perp' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT}$$

$$U' R_{0T} - U' U_\perp \alpha_{11} A_1' R_{PT} + U' \Theta R_E - U' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} + \\ + U' U_\perp \alpha_{11} A_1' \Im_{PP} \Im_{PP.\hat{\Pi}_2} R_{PT} - U' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT}$$

or

$$U_\perp' R_{0T} - \alpha_{11} A_1' R_{PT} + U_\perp' \Theta R_E - U_\perp' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} + \\ + \alpha_{11} A_1' \Im_{PP} \Im_{PP.\hat{\Pi}_2} R_{PT} - U_\perp' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT} \tag{4.35}$$

$$U' R_{0T} + U' \Theta R_E - U' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} - U' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT} \tag{4.36}$$

because $U_\perp{}' U_\perp = I_{k_1}$ and $U' U_\perp = 0$. The likelihood function factors as

$$L(\omega) = \prod_{T=1}^{N} f\left(U' E_T ; \omega_{h_2}\right) \prod_{T=1}^{N} f\left(U_\perp' E_T \middle| U' E_T ; \omega_{h_1}\right) \tag{4.37}$$

where $\omega = (\alpha_{11}, A_1, \Theta, \Omega_E)$, $\omega_{h_1} = (\alpha_{11}, A_1, \Omega_E)$ and $\omega_{h_2} = (\Theta, \Omega_E)$.

The factor of the likelihood function corresponding to the marginal distribution of the variable in Equation (4.36) is given by

$$L\left(U' E_T ; \omega_{h_2}\right) \propto \left|\Omega_{UU}\right|^{-N/2} \exp\left\{-\frac{1}{2} \times \right. \\ \times \sum_{T=1}^{N} \left(U' R_{0T} + U' \Theta R_E - U' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} - U' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT}\right)' \Omega_{UU}^{-1} \times \\ \times \left(U' R_{0T} + U' \Theta R_E - U' \Im_{0P} \Im_{PP.\hat{\Pi}_2} R_{PT} - U' \Theta H_{EP} \Im_{PP.\hat{\Pi}_2} R_{PT}\right) \tag{4.38}$$

where $\Omega_{UU} = U' \Omega_E U$. The MLE of Θ is calculated as

$$\frac{\partial \ln L\left(U'E_T;\omega_{h_2}\right)}{\partial\Theta} = UU'H_{0E} - UU'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} - UU'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} + \\ + UU'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE} + UU'\Theta H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE} + \\ + UU'\Theta H_{EE} - UU'\Theta H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE} - UU'\Theta H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}$$

Since

$$\Im_{PP.\hat{\Pi}_2} = \Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2} \\ = \hat{\Pi}_2'\left(\hat{\Pi}_2\Im_{PP}\hat{\Pi}_2'\right)^{-1}\hat{\Pi}_2\Im_{PP}\hat{\Pi}_2'\left(\hat{\Pi}_2\Im_{PP}\hat{\Pi}_2'\right)^{-1}\hat{\Pi}_2 = \hat{\Pi}_2'\left(\hat{\Pi}_2\Im_{PP}\hat{\Pi}_2'\right)^{-1}\hat{\Pi}_2,$$

$$\frac{\partial \ln L\left(U'E_T;\omega_{h_2}\right)}{\partial\Theta} = UU'H_{0E} - UU'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} + UU'\Theta H_{EE} - UU'\Theta H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE} = \mathbf{0}.$$

Hence,

$$UU'\left(H_{0E} - \Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} + \Theta H_{EE} - \Theta H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\right) = \mathbf{0}.$$

The MLE of Θ is given by

$$\hat{\Theta} = \left(\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} - H_{0E}\right)H_{EE.\hat{\Pi}_2}^{-1} \qquad (4.39)$$

where $H_{EE.\hat{\Pi}_2} = H_{EE} - H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}$.

When we replace Θ in Equation (4.36) by $\hat{\Theta}$ in Equation (4.39), we have

$$U'R_{0T} + U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}H_{EE.\hat{\Pi}_2}^{-1}R_E - U'H_{0E}H_{EE.\hat{\Pi}_2}^{-1}R_E - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT} \\ - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}R_{PT} + U'F_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}.$$

The maximum likelihood estimator of Ω_{UU} is calculated as

$$\frac{\partial \ln L\left(U'E_T;\omega_{h_1}\right)}{\partial\Omega_{UU}} = \mathbf{0}.$$

$$\frac{\partial \ln L\left(U'E_T;\omega_{k_1}\right)}{\partial \Omega_{UU}} = U'\left(\Im_{00}-F_{00}\right)U - U\left(\Im_{0P}-F_{0P}\right)\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{P0}U +$$
$$+U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{P0}U + U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}H^{-1}_{EE.\hat{\Pi}_2}H_{EE}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}H^{-1}_{EE.\hat{\Pi}_2}H_{EE}H^{-1}_{EE.\hat{\Pi}_2}H_{E0}U - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}F_{P0}U$$
$$-U'F_{00}U - U'H_{0E}H^{-1}_{EE.\hat{\Pi}_2}H_{EE}H^{-1}_{EE.\hat{\Pi}_2}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$+U'H_{0E}H^{-1}_{EE.\hat{\Pi}_2}H_{EE}H^{-1}_{EE.\hat{\Pi}_2}H_{E0}U + U'F_{0P}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U$$
$$+U'F_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$+U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U + U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U$$
$$-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$+U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U$$
$$+U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$+U'F_{0P}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U + U'F_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$
$$-U'F_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)U - U'F_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U$$

where $F_{00} = H_{0E}H^{-1}_{EE.\hat{\Pi}_2}H_{E0}$, $F_{0P} = H_{0E}H^{-1}_{EE.\hat{\Pi}_2}H_{EP}$, $F_{P0} = H_{PE}H^{-1}_{EE.\hat{\Pi}_2}H_{E0}$ and $F_{PP} = H_{PE}H^{-1}_{EE.\hat{\Pi}_2}H_{EP}$.

$$\hat{\Omega}_{UU} = U'\Im_{00.\hat{\Pi}_2}U \tag{4.40}$$

where $\Im_{00.\hat{\Pi}_2} = \left(\Im_{00}-F_{00}\right)-\left(\Im_{0P}-F_{0P}\right)\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)$. The maximized likelihood function is given by

$$L^{-2/N}_{\max U} = \left|U'\Im_{00.\hat{\Pi}_2}U\right| = \left|U\left[\left(\Im_{00}-F_{00}\right)-\left(\Im_{0P}-F_{0P}\right)\Im_{PP.\hat{\Pi}_2}\left(\Im_{P0}-F_{P0}\right)\right]U'\right|. \tag{4.41}$$

The other factor of the likelihood function corresponds to the conditional distribution of $U'_{\perp}E_T \left| U'E_T \right.$ and it is given by

$$
\begin{aligned}
&\left|\Omega_{U_\perp U_\perp .U}\right|^{-N/2} \exp\left\{-\frac{1}{2}\sum_{T=1}^{N}\Big(U'_\perp R_{0T} - \alpha_{11}A'_1 R_{PT} + U'_\perp \hat{\Theta} R_E - U'_\perp \Im_{0P}\Im_{PP.\hat{\Pi}_2} R_{PT} + \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2} R_{PT} - \right.\\
&\quad -U'_\perp \hat{\Theta} H_{EP}\Im_{PP.\hat{\Pi}_2} R_{PT} - W_1\left(U'R_{0T} + U'\hat{\Theta}R_E - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2} R_{PT} - U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2} R_{PT}\right)\Big)'\\
&\quad \times\Omega^{-1}_{U_\perp U_\perp .U}\sum_{T=1}^{N}\Big(U'_\perp R_{0T} - \alpha_{11}A'_1 R_{PT} + U'_\perp \hat{\Theta} R_E - U'_\perp \Im_{0P}\Im_{PP.\hat{\Pi}_2} R_{PT} + \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2} R_{PT} - \\
&\quad \left. -U'_\perp \hat{\Theta} H_{EP}\Im_{PP.\hat{\Pi}_2} R_{PT} - W_1\left(U'R_{0T} + U'\hat{\Theta}R_E - U'\Im_{0P}\Im_{PP.\hat{\Pi}_2} R_{PT} - U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2} R_{PT}\right)\Big)\right.
\end{aligned}
\tag{4.42}
$$

$W_1 = \Omega_{U_\perp U_\perp .U}\Omega^{-1}_{UU}$. The estimator of W_1 is found for fixed α_{11} and A_1.

$$
\begin{aligned}
\frac{\partial \ln L}{\partial W_1} &= U'_\perp\Im_{00}U - \alpha_{11}A'_1\Im_{P0}U + U'_\perp\hat{\Theta}H_{E0}U - U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&+\alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U - U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&U'_\perp H_{0E}\hat{\Theta}'U - \alpha_{11}A'_1 H_{PE}\hat{\Theta}'U + U'_\perp\hat{\Theta}H_{EE}\hat{\Theta}'U\\
&-U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U + \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U - \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&-U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&-U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - \alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&+U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - U'_\perp\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&+\alpha_{11}A'_1\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - U'_\perp\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&-W_1U'\Im_{00}U - W_1U'\hat{\Theta}H_{E0}U + W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&-W_1U'H_{0E}\hat{\Theta}'U - W_1U'\hat{\Theta}H_{EE}\hat{\Theta}'U + W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&+W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U - W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&+W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U + W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\\
&-W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U - W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\\
&+W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U + W_1U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U = 0
\end{aligned}
$$

where $\hat{\Theta} = \left(\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE} - H_{0E}\right)H^{-1}_{EE.\hat{\Pi}_2}$. After eliminating some matrices, we have

$$\begin{aligned}
&U_{\perp}'(\Im_{00}-F_{00})U-\alpha_{11}A_1'(\Im_{P0}-F_{P0})U \\
&-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}(\Im_{P0}-F_{P0})U-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U \\
&+U_{\perp}'F_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{0P}U \\
&+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}(\Im_{P0}-F_{P0})U-\alpha_{11}A_1'F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{0P}U \\
&-W_1U'(\Im_{00}-F_{00})U+W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}(\Im_{P0}-F_{P0})U \\
&+W_1U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U-W_1U'F_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U=\mathbf{0}.
\end{aligned}$$

Then, the MLE of W_1 is given by

$$\hat{W}_1=\left\{U_{\perp}'\Im_{00.\hat{\Theta}}U-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\right\}\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1} \tag{4.43}$$

where

$$\begin{aligned}
\Im_{00.\hat{\Theta}}&=(\Im_{00}-F_{00})-\Im_{0P}\Im_{PP.\hat{\Pi}_2}(\Im_{P0}-F_{P0})-\Im_{0P}\Im_{PP.\hat{\Pi}_2}F_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}-F_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0} \\
&=\Im_{00}-\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}-\hat{\Theta}H_{EE.\hat{\Pi}_2}\hat{\Theta}'.
\end{aligned}$$

When we replace $\hat{W}_1$ in Equation (4.43) by W in

$$\begin{aligned}
&U_{\perp}'R_{0T}-\alpha_{11}A_1'R_{PT}+U_{\perp}'\hat{\Theta}R_E-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}R_{PT}-U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}R_{PT} \\
&\quad -W_1\left(U'R_{0T}+U'\hat{\Theta}R_E-U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}-U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}R_{PT}\right)
\end{aligned}$$

we obtain

$$\begin{aligned}
&U_{\perp}'R_{0T}-\alpha_{11}A_1'R_{PT}+U_{\perp}'\hat{\Theta}R_E-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}R_{PT}- \\
&U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}R_{PT}--U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'R_{0T}-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}R_E+ \\
&+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}R_{PT}+ \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\right\}\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'R_{0T}+ \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\right\}\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}R_E- \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\right\}\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}R_{PT}- \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\right\}\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}R_{PT}.
\end{aligned} \tag{4.44}$$

The derivative of log-likelihood with respect to α_{11} is written as

$$\begin{gathered}
U_\perp'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-U_\perp'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
U_\perp'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
+U_\perp'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
+U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1
\end{gathered}$$

$$\begin{gathered}
+U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
U_\perp'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
+U_\perp'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1
\end{gathered}$$

$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$-U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$-U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$-U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$-U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1$$

$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]$$

$$-U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]$$

$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times$$

$$\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]$$

$$
\begin{gathered}
+U_{\perp}'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right] \\
+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right] \\
+U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right] \\
-U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right] \\
-U_{\perp}'\Im_{00.\dot{\Theta}}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right] \\
+\alpha_{11}A_1'\Im_{PP}A_1 \\
-\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}A_1 \\
-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}A_1 \\
+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1 \\
-\alpha_{11}A_1'\Im_{P0}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{00}U\times \\
\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1
\end{gathered}
$$

$$
\begin{aligned}
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1 \\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times \\
&\quad\times\left(U'\Im_{00.\dot{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&+\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&-\alpha_{11}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&+\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\times\\
&\quad\times\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1=\mathbf{0}.
\end{aligned}
$$

By using the relations

$$\Im_{00.\hat{\Theta}}=\Im_{00}-\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}-\hat{\Theta}H_{EE.\hat{\Pi}_2}\hat{\Theta}',$$

$$H_{EE.\hat{\Pi}_2}=H_{EE}-H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE},$$

and

$$\hat{\Theta}=\left(\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}-H_{0E}\right)H_{EE.\hat{\Pi}_2}^{-1},$$

the derivative can be simplified as

$$
\begin{aligned}
&-U_\perp'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&+U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\\
&+\alpha_{11}A_1'\left(\Im_{PP}-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right)A_1\\
&-\alpha_{11}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1=\mathbf{0}.
\end{aligned}
$$

Hence, the MLE of α_{11} for fixed A_1 is given by

$$\hat{\alpha}_{11}\left(A_1\right)=U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1} \tag{4.45}$$

where

$$\Im_{0P.\hat{\Theta}}=\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]-$$
$$-\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]$$

and

$$\Im_{PP.\hat{\Theta}}=\left(\Im_{PP}-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right)-$$
$$-\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right].$$

The MLE of $\Omega_{U_{\perp}U_{\perp}.U}$ is given as

$$\hat{\Omega}_{U_{\perp}U_{\perp}.U}=U_{\perp}'\Im_{00}U_{\perp}-U_{\perp}'\Im_{0P}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'H_{0E}\hat{\Theta}'U_{\perp}-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}-U_{\perp}'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$
\begin{gathered}
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0}U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'H_{PE}\hat{\Theta}'U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times \\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times \\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times \\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times \\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
U_\perp'\hat{\Theta}H_{E0}U_\perp-U_\perp'\hat{\Theta}H_{EP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
+U_\perp'\hat{\Theta}H_{EE}\hat{\Theta}'U_\perp-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp \\
+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
-U_\perp'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp-U_\perp'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp \\
+U_\perp'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
+U_\perp'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \\
-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp
\end{gathered}
$$

$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$
\begin{aligned}
&+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp\\
&+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp\\
&+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp\\
&+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&-U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp\\
&+U_\perp'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
&\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp
\end{aligned}
$$

$$
\begin{gathered}
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times\\
\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}\\
-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}\\
+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}
\end{gathered}
$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'H_{0E}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$
$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$-U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{E0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_{\perp}$$

$$+U_{\perp}'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$

$$\times\Im_{0P}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\times$$

$$\times\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\times$$

$$\times\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp$$

$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$

$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$

$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$
$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$
$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\times$$
$$\times\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp$$
$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$
$$-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\times$$
$$\times\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$
$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\times$$
$$\times\Im_{PP.\hat{\Pi}_2}\Im_{P0}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp$$
$$+U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\left[I-\Im_{PP}\Im_{PP.\hat{\Pi}_2}\right]\left(\Im_{P0}+H_{PE}\hat{\Theta}'\right)U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\hat{\Theta}H_{EP}\Im_{PP.\hat{\Pi}_2}\Im_{PP}\times$$
$$\times\Im_{PP.\hat{\Pi}_2}H_{PE}\hat{\Theta}'U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\left(\Im_{0P}+\hat{\Theta}H_{EP}\right)\left[I-\Im_{PP.\hat{\Pi}_2}\Im_{PP}\right]A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp.$$

After some simplification, the MLE of $\Omega_{U_\perp U_\perp.U}$ is given by

$$\hat{\Omega}_{U_\perp U_\perp.U}\left(A_1\right)=U_\perp'\Im_{00.U_\perp}U_\perp-U_\perp'\Im_{0P.\hat{\Theta}}A_1\left(A_1'\Im_{PP.\hat{\Theta}}A_1\right)^{-1}A_1'\Im_{P0.\hat{\Theta}}U_\perp \tag{4.46}$$

where

$$\Im_{00.U_\perp}=U_\perp'\Im_{00}U_\perp-U_\perp'\Im_{0P}\Im_{PP.\hat{\Pi}_2}\Im_{P0}U_\perp-$$
$$-U_\perp'\hat{\Theta}H_{EE.\hat{\Pi}_2}\hat{\Theta}'U_\perp-U_\perp'\Im_{00.\hat{\Theta}}U\left(U'\Im_{00.\hat{\Theta}}U\right)^{-1}U'\Im_{00.\hat{\Theta}}U_\perp.$$

By following the steps given in Section 3.4.1, the MLE of A_1, $\hat{A}_{1,i+1}$, are the first h_l eigenvectors of

$$\Im_{P0.\hat{\Theta}}U_\perp\left(U_\perp'\Im_{00.U_\perp}U_\perp\right)^{-1}U_\perp'\Im_{0P.\hat{\Theta}}\Im_{PP.\hat{\Theta}}^{-1}.$$

Finally, by taking into account the restriction $U'\eta V=0$ for the given values of

$\hat{\alpha}_{11.i+1}$, $\hat{A}_{1.i+1}$, $\hat{\alpha}_{2.i}$ and $\hat{A}_{22.i}$, (i+1)th step estimate for η is derived by the Lagrangian function

$$\ln L(\eta,\Omega_E,\Xi) \propto -\frac{N}{2}\ln|\Omega_E| - \left\{ -tr\left[\Omega_E^{-1}\left(\hat{Z}_0 - \eta Z_1\right)\left(\hat{Z}_0 - \eta Z_1\right)' \right]/2 \right\} - tr[\Xi' U' \eta V]$$

(4.47)

where $\hat{Z}_0 = Z_0 - (U_\perp \hat{\alpha}_{11}\hat{A}_1' + \hat{\alpha}_2 \hat{A}_{22}' U')Z_P + \hat{\Theta}E$ and Ξ is a matrix Lagrange multipliers (Spanos, 1986). Therefore,

$$\frac{\partial \ln L(\eta,\Omega_E,\Xi)}{\partial \eta} = \left\{ -\left[\Omega_E^{-1}\left(\hat{Z}_0 Z_1' - \eta Z_1 Z_1'\right)\right]\right\} - U\Xi V' = 0, \quad (4.48)$$

$$\frac{\partial \ln L(\eta,\Omega_E,\Xi)}{\partial \Omega_E} = \frac{N}{2}\Omega_E - \frac{1}{2}\left[\left(\hat{Z}_0 - \eta Z_1\right)\left(\hat{Z}_0 - \eta Z_1\right)'\right] = 0, \quad (4.49)$$

$$\frac{\partial \ln L(\eta,\Omega_E,\Xi)}{\partial \Xi} = -U'\eta V = 0. \quad (4.50)$$

From Equation (4.48) we can obtain

$$\left[\Omega_E^{-1}(\hat{\eta} - \eta)(Z_1 Z_1')\right] = U\Xi V' \quad (4.51)$$

where $\hat{\eta} = \hat{Z}_0 Z_1'(Z_1 Z_1')^{-1}$. Pre-multiplying both sides of Equation (4.51) by Ω_E gives

$$\left[(\hat{\eta} - \eta)(Z_1 Z_1')\right] = \Omega_E U \Xi V', \quad (4.52)$$

and pre-multiplying both sides of Equation (4.52) by U'

$$\left[(U'\hat{\eta} - U'\eta)(Z_1 Z_1')\right] = U'\Omega_E U \Xi V'. \quad (4.53)$$

By post-multiplying by $(Z_1 Z_1')^{-1}$, we can obtain

$$\left[(U'\hat{\eta} - U'\eta)\right] = U'\Omega_E U \Xi V'(Z_1 Z_1')^{-1}. \quad (4.54)$$

Equation (4.54) can be written as

$$\left[\left(U'\Omega_E U\right)^{-1}\left(U'\hat{\eta}-U'\eta\right)\right]=\Xi V'\left(Z_1Z_1'\right)^{-1}. \tag{4.55}$$

Post-multiplying both sides of Equation (4.55) by V

$$\left[\left(U'\Omega_E U\right)^{-1}\left(U'\hat{\eta}V-U'\eta V\right)\right]=\Xi V'\left(Z_1Z_1'\right)^{-1}V. \tag{4.56}$$

By Equation (4.50)

$$\left(U'\Omega_E U\right)^{-1}\left(U'\hat{\eta}V\right)=\Xi V'\left(Z_1Z_1'\right)^{-1}V. \tag{4.57}$$

Hence, the Lagrange multiplier is given by

$$\Xi=\left(U'\Omega_E U\right)^{-1}\left(U'\hat{\eta}V\right)\left(V'\left(Z_1Z_1'\right)^{-1}V\right)^{-1}. \tag{4.58}$$

Replacing Equation (4.58) in Equation (4.52) gives

$$\left[\left(\hat{\eta}-\eta\right)\left(Z_1Z_1'\right)\right]=\Omega_E U\left(U'\Omega_E U\right)^{-1}\left(U'\hat{\eta}V\right)\left(V'\left(Z_1Z_1'\right)^{-1}V\right)^{-1}V'.$$

So, the MLE of η under the restriction $U'\eta V=0$ is given by

$$\tilde{\eta}_{i+1}=\hat{\eta}-\hat{\Omega}_E U\left(U'\hat{\Omega}_E U\right)^{-1}\left(U'\hat{\eta}V\right)\left(V'\left(Z_1Z_1'\right)^{-1}V\right)^{-1}V'\left(Z_1Z_1'\right)^{-1} \tag{4.59}$$

where $\hat{\Omega}_E=\frac{1}{N}\left[\left(\hat{Z}_0-\hat{\eta}Z_1\right)\left(\hat{Z}_0-\hat{\eta}Z_1\right)'\right]$. The MLE of the error variance matrix is

$$\tilde{\Omega}_{E,i+1}=\hat{\Omega}_E+\frac{1}{N}\left(\tilde{\eta}_{i+1}-\hat{\eta}\right)\left(Z_1Z_1'\right)\left(\tilde{\eta}_{i+1}-\hat{\eta}\right)' \tag{4.60}$$

where $\hat{\Omega}_E=U_\perp'\left(\Im_{00.\hat{\Pi}_2}-F_{00.\hat{\Pi}_2}\right)U_\perp-U_\perp'\left(\Im_{0P.\hat{\Pi}_2}-F_{0P.\hat{\Pi}_2}\right)\hat{A}_1\hat{A}_1'\left(\Im_{P0.\hat{\Pi}_2}-F_{P0.\hat{\Pi}_2}\right)U_\perp$ (see Richard (1979)).

The maximized likelihood is proportional to $\left|\tilde{\Omega}_{E,i+1}\right|^{-N/2}$.

$$\left|\tilde{\Omega}_{E,i+1}\right| = \left|\hat{\Omega}_E + \frac{1}{N}\left(\hat{\Omega}_E U\left(U'\hat{\Omega}_E U\right)^{-1}(U'\hat{\eta}V)\left(V'(Z_1Z_1')^{-1}V\right)^{-1}V'(Z_1Z_1')^{-1}\right)\times\right.$$
$$\left.\times(Z_1Z_1')\left(\hat{\Omega}_E U\left(U'\hat{\Omega}_E U\right)^{-1}(U'\hat{\eta}V)\left(V'(Z_1Z_1')^{-1}V\right)^{-1}V'(Z_1Z_1')^{-1}\right)'\right|$$

$$\left|\tilde{\Omega}_{E,i+1}\right| = \left|\hat{\Omega}_E\right|\left|I + \frac{1}{N}\left(U\left(U'\hat{\Omega}_E U\right)^{-1}(U'\hat{\eta}V)\left(V'(Z_1Z_1')^{-1}V\right)^{-1}V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\Omega}_E\right)\right|$$

$$\left|\tilde{\Omega}_{E,i+1}\right| = \left|\hat{\Omega}_E\right|\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\Omega}_E U\left(U'\hat{\Omega}_E U\right)^{-1}(U'\hat{\eta}V)\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right|$$

$$\left|\tilde{\Omega}_{E,i+1}\right| = \left|\hat{\Omega}_E\right|\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\eta}V\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right|$$

Therefore, the maximized likelihood is given by

$$L_{\max,i+1}^{-2/N} = \left|U_\perp'\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right|$$
$$\left|I - \hat{A}_{1,i+1}'\left(\Im_{P0.\hat{\Pi}_2} - F_{P0.\hat{\Pi}_2}\right)U_\perp\left(U_\perp'\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right)^{-1}U_\perp'\left(\Im_{0P.\hat{\Pi}_2} - F_{0P.\hat{\Pi}_2}\right)\hat{A}_{1,i+1}\right| \quad (4.61)$$
$$\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\eta}V\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right|.$$

By using the relationship between eigenvalues and eigenvectors, we obtain

$$\left(\Im_{P0.\hat{\Pi}_2} - F_{P0.\hat{\Pi}_2}\right)U_\perp\left(U_\perp'\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right)^{-1}U_\perp'\left(\Im_{0P.\hat{\Pi}_2} - F_{0P.\hat{\Pi}_2}\right)\hat{A}_{1,i+1} =$$
$$= \left(\Im_{PP.\hat{\Pi}_2} - F_{PP.\hat{\Pi}_2}\right)\hat{A}_{1,i+1}\mathrm{T}_{A_{1,i+1}}$$

where $\mathrm{T}_{A_{1,i+1}}$ denotes the diagonal matrix of ordered eigenvalues and $\hat{A}_{1,i+1}$ is the matrix of the corresponding eigenvectors. Hence Equation (4.61) given as

$$L_{\max,i+1}^{-2/N} = \left|U_\perp'\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right|\left|I_{k_1} - \hat{A}_{1,i+1}'\left(\Im_{PP.\hat{\Pi}_2} - F_{PP.\hat{\Pi}_2}\right)\hat{A}_{1,i+1}\mathrm{T}_{A_{1,i+1}}\right|$$
$$\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\eta}V\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right|,$$

and since $\hat{A}'_{1,i+1}\left(\Im_{PP.\hat{\Pi}_2} - F_{PP.\hat{\Pi}_2}\right)\hat{A}_{1,i+1} = I_{h_1}$,

$$L^{-2/N}_{\max,i+1} = \left|U'_\perp\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right|\left|I - T_{A_1}\right|\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\eta}V\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right|.$$

Therefore, the maximized likelihood function is given by

$$L^{-2/N}_{\max,i+1} = \left|U'_\perp\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right|\left\{\prod_{j=1}^{h_1}(1-\hat{\lambda}_{j,i+1})\right\}\times \\ \times\left|I + \frac{1}{N}\left(V'\hat{\eta}'U\left(U'\hat{\Omega}_E U\right)^{-1}U'\hat{\eta}V\left(V'(Z_1Z_1')^{-1}V\right)^{-1}\right)\right| \tag{4.62}$$

where $\hat{\lambda}_{j,i+1}$ s are ordered eigenvalues $1 > \hat{\lambda}_{1,i+1} > \ldots > \hat{\lambda}_{k,i+1} > 0$ of $\left(\Im_{P0.\hat{\Pi}_2} - F_{P0.\hat{\Pi}_2}\right)U_\perp\left(U'_\perp\left(\Im_{00.\hat{\Pi}_2} - F_{00.\hat{\Pi}_2}\right)U_\perp\right)^{-1}U'_\perp\left(\Im_{0P.\hat{\Pi}_2} - F_{0P.\hat{\Pi}_2}\right)\left(\Im_{PP.\hat{\Pi}_2} - F_{PP.\hat{\Pi}_2}\right)^{-1}$.

Given $\hat{\alpha}_{11,i+1}, \hat{A}_{1,i+1}, \hat{\eta}_{i+1}$ and $\hat{\Theta}_{i+1}$, (i+2)-th step estimates for $\hat{\alpha}_{2,i+2}, \hat{A}_{22,i+2}$ and $\hat{\Omega}_{E,i+2}$ are obtained from the following conditional likelihood function

$$\frac{\partial \ln L\left(\alpha_{i+2}, A_{22,i+2}, \Omega_{E,i+2}\right)}{\partial \nabla} \propto -\frac{N}{2}\ln\left|\Omega_{E,i+2}\right| - \{\mathrm{tr}\left[\Omega^{-1}_{E,i+2}\left(Z_{0T} - \tilde{\eta}Z_{1T} - \hat{\Pi}_1 Z_{PT} - \right.\right. \\ -\alpha_2 A'_{22}U'Z_{PT} + \hat{\Theta}E - \nabla\left(\hat{\Pi}_1 Z_{PT} - \hat{\Theta}E - \tilde{\eta}Z_{1T}\right)\left(Z_{0T} - \tilde{\eta}Z_{1T} - \hat{\Pi}_1 Z_{PT} - \right. \\ \left. -\alpha_2 A'_{22}U'Z_{PT} + \hat{\Theta}E - \nabla\left(\hat{\Pi}_1 Z_{PT} - \hat{\Theta}E - \tilde{\eta}Z_{1T}\right)'/2 \tag{4.63}$$

where $\hat{\Pi}_1 = U_\perp \hat{\alpha}_{1,i}\hat{A}'_{1,i}$ and $\nabla = \Omega_{E,i+2,i+1}\Omega^{-1}_{E,i+1}$. The analysis of Equation (4.61) is similar to the reduced rank regression of $Z_{0T} + \alpha_2 A'_{22}U'Z_{PT}$ on $\tilde{\eta}Z_{1T} + \hat{\Pi}_1 Z_{PT} - \hat{\Theta}E$ corrected for $\nabla\left(\tilde{\eta}Z_{1T} + \hat{\Pi}_1 Z_{PT} - \hat{\Theta}E\right)$. Therefore, the estimator of ∇ is calculated for a fixed α_2 and A_{22} as

$$\frac{\partial \ln L\left(\alpha_{i+2}, A_{22,i+2}, \Omega_{E,i+2}\right)}{\partial \nabla} = -\mathrm{M}_{01}\tilde{\eta}' - \mathrm{M}_{0P}\hat{\Pi}_1' + \aleph_{0E}\hat{\Theta}' + \tilde{\eta}\mathrm{M}_{11}\tilde{\eta}' - \tilde{\eta}\mathrm{M}_{1P}\hat{\Pi}_1' + \tilde{\eta}\aleph_{1E}\hat{\Theta}' +$$
$$+\hat{\Pi}_1\mathrm{M}_{P1}\tilde{\eta}' - \hat{\Pi}_1\mathrm{M}_{PP}\hat{\Pi}_1' + \hat{\Pi}_1\aleph_{PE}\hat{\Theta}' + \alpha_2 A_{22}' U'\mathrm{M}_{P1}\tilde{\eta}' - \alpha_2 A_{22}' U'\mathrm{M}_{PP}\hat{\Pi}_1' +$$
$$+\alpha_2 A_{22}' U'\aleph_{PE}\hat{\Theta}' - -\hat{\Theta}\aleph_{E1}\tilde{\eta}' - \hat{\Theta}\aleph_{EP}\hat{\Pi}_1' + \hat{\Theta}\aleph_{EE}\hat{\Theta}' +$$
$$+\nabla\left(\tilde{\eta}\mathrm{M}_{11}\tilde{\eta}' - \tilde{\eta}\mathrm{M}_{1P}\hat{\Pi}_1' + \tilde{\eta}\aleph_{1E}\hat{\Theta}' + \hat{\Pi}_1\mathrm{M}_{P1}\tilde{\eta}' - \hat{\Pi}_1\mathrm{M}_{PP}\hat{\Pi}_1' + \hat{\Pi}_1\aleph_{PE}\hat{\Theta}' -\right.$$
$$\left.-\hat{\Theta}\aleph_{E1}\tilde{\eta}' - \hat{\Theta}\aleph_{EP}\hat{\Pi}_1' + \hat{\Theta}\aleph_{EE}\hat{\Theta}'\right) = 0$$

$$\hat{\nabla}\left(\alpha_2, A_{22}\right) = \left\{\mathrm{M}_{01}\tilde{\eta}' - \mathrm{M}_{0P}\hat{\Pi}_1' - \aleph_{0E}\hat{\Theta}' + \alpha_2 A_{22}' U'\left(\mathrm{M}_{P1}\tilde{\eta}' - \mathrm{M}_{PP}\hat{\Pi}_1' + \aleph_{PE}\hat{\Theta}'\right)\right\}\Sigma^{-1} - I \tag{4.64}$$

where

$$\Sigma = \left(\tilde{\eta}\mathrm{M}_{11}\tilde{\eta}' - \tilde{\eta}\mathrm{M}_{1P}\hat{\Pi}_1' + \tilde{\eta}\aleph_{1E}\hat{\Theta}' + \hat{\Pi}_1\mathrm{M}_{P1}\tilde{\eta}' - \hat{\Pi}_1\mathrm{M}_{PP}\hat{\Pi}_1' + \hat{\Pi}_1\aleph_{PE}\hat{\Theta}' -\right.$$
$$\left.-\hat{\Theta}\aleph_{E1}\tilde{\eta}' - \hat{\Theta}\aleph_{EP}\hat{\Pi}_1' + \hat{\Theta}\aleph_{EE}\hat{\Theta}'\right)$$

Replacing ∇ by Equation (4.64) the following function which is the part of the log-likelihood function given in Equation (4.63)

$$Z_{0T} - \tilde{\eta}Z_{1T} - \hat{\Pi}_1 Z_{PT} - \alpha_2 A_{22}' U' Z_{PT} + \hat{\Theta}E - \nabla\left(\tilde{\eta}Z_{1T} + \hat{\Pi}_1 Z_{PT} - \hat{\Theta}E\right)$$

gives

$$\Re_{0T} - \alpha_2 A_{22}' U' \Re_{PT} \tag{4.65}$$

where $\Re_{0T} = Z_{0T} - \left(\mathrm{M}_{01}\tilde{\eta}' - \mathrm{M}_{0P}\hat{\Pi}_1' - \aleph_{0E}\hat{\Theta}'\right)\Sigma^{-1}\left(\tilde{\eta}Z_{1T} + \hat{\Pi}_1 Z_{PT} - \hat{\Theta}E\right)$ and

$\Re_{PT} = Z_{PT} - \left(\mathrm{M}_{P1}\tilde{\eta}' - \mathrm{M}_{PP}\hat{\Pi}_1' + \aleph_{PE}\hat{\Theta}'\right)\Sigma^{-1}\left(\tilde{\eta}Z_{1T} + \hat{\Pi}_1 Z_{PT} - \hat{\Theta}E\right)$.

For a fixed A_{22}, the MLE of α_2 is obtained as

$$\frac{\partial \ln L\left(\alpha_{i+2}, A_{22,i+2}, \Omega_{E,i+2}\right)}{\partial \alpha_2} = -\Re_{0T}\Re_{PT}' U A_{22} + \alpha_{22} A_{22}' U' \Re_{PT}\Re_{PT}' A_{22} = 0$$

$$\hat{\alpha}_{2,i+2}(A_{22}) = \Re_{0T}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}. \tag{4.66}$$

By replacing α_2 by $\hat{\alpha}_{2,i+2}(A_{22})$ in Equation (4.65), we obtain

$$\Re_{0T} - \Re_{0T}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}A'_{22}U'\Re_{PT}.$$

For a fixed A_{22}, the MLE of $\Omega_{E,i+2}$ can be calculated as

$$\frac{\partial \ln L\left(\alpha_{i+2}, A_{22,i+2}, \Omega_{E,i+2}\right)}{\partial \Omega_{E,i+2}} = \Re_{0T}\Re'_{0T} - 2\Re_{0T}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}\times$$
$$\times A'_{22}U'\Re_{PT}\Re'_{0T} + \Re_{0T}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}\times$$
$$\times A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}A'_{22}U'\Re_{PT}\Re'_{0T} = 0.$$

Hence, the MLE of $\Omega_{E,i+2}$ for a fixed value of A_{22} is given by

$$\hat{\Omega}_{E,i+2} = \Re_{0T}\Re'_{0T} - \Re_{0T}\Re'_{PT}UA_{22}\left(A'_{22}U'\Re_{PT}\Re'_{PT}UA_{22}\right)^{-1}A'_{22}U'\Re_{PT}\Re'_{0T}. \tag{4.67}$$

By following the steps given in Section 3.4.1, the MLE of A_{22}, $\hat{A}_{22,i+2}$ are the first h_2 eigenvectors of $U'\Re_{PT}\Re'_{0T}\left(\Re_{0T}\Re'_{0T}\right)^{-1}\Re_{0T}\Re'_{PT}U\left(U'\Re_{PT}\Re'_{PT}U\right)^{-1}$. The maximized likelihood is proportional to $\left|\hat{\Omega}_{E,i+2}\right|^{-N/2}$. Therefore, the maximized likelihood is given by

$$L^{-2/N}_{\max,i+2} = \left|\Re_{0T}\Re'_{0T}\right|\left|I - \hat{A}'_{22,i+1}U'\Re_{PT}\Re'_{0T}\left(\Re_{0T}\Re'_{0T}\right)^{-1}\Re_{0T}\Re'_{PT}U\hat{A}_{22,i+1}\right|. \tag{4.68}$$

By using the relationship between eigenvalues and eigenvectors, we obtain

$$U'\Re_{PT}\Re'_{0T}\left(\Re_{0T}\Re'_{0T}\right)^{-1}\Re_{0T}\Re'_{PT}U\hat{A}_{22,i+1} = U'\Re_{PT}\Re'_{PT}U\hat{A}_{22,i+2}\mathrm{T}_{A_{22,i+2}}$$

where $\mathrm{T}_{A_{22,i+2}}$ denotes the diagonal matrix of ordered eigenvalues and $\hat{A}_{22,i+2}$ is the matrix of the corresponding eigenvectors. Hence, Equation (4.61) becomes

$$L^{-2/N}_{\max,i+2} = \left|\Re_{0T}\Re'_{0T}\right|\left|I - \hat{A}'_{22,i+1}U'\Re_{PT}\Re'_{PT}U\hat{A}_{22,i+2}\mathrm{T}_{A_{22,i+2}}\right|$$

and since $\hat{A}'_{22,i+2}U'\Re_{PT}\Re'_{PT}U\hat{A}_{22,i+2} = I$,

$$L^{-2/N}_{\max,i+2} = \left|\Re_{0T}\Re'_{0T}\right|\left|I - T_{A_{22,i+2}}\right|.$$

Therefore, the maximized likelihood function is given by

$$L^{-2/N}_{\max,i+2} = \left|\Re_{0T}\Re'_{0T}\right|\prod_{j=1}^{h_2}(1-\hat{\lambda}_{j,i+2}) \tag{4.69}$$

where $\hat{\lambda}_{j,i+2}$'s are ordered eigenvalues $1 > \hat{\lambda}_{1,i+2} > \ldots > \hat{\lambda}_{k,i+2} > 0$ of $U'\Re_{PT}\Re'_{0T}\left(\Re_{0T}\Re'_{0T}\right)^{-1}\Re_{0T}\Re'_{PT}U\left(U'\Re_{PT}\Re'_{PT}U\right)^{-1}$.

To be able to obtain the MLEs of all the parameters, one should obtain Equations (4.62) and (4.68) until the difference between the values in equations are very small.

The likelihood ratio non-causality test statistic is given by

$$-2\ln\frac{\max\limits_{H_0(h,h_1,h_2)} L\left[\eta,\Pi_{AG},\Theta;X_1,\cdots,X_N\right]}{\max\limits_{H_A(h)} L\left[\eta,\Pi_{AG},\Theta;X_1,\cdots,X_N\right]} \tag{4.70}$$

where $\max\limits_{H_0(h,h_1,h_2)} L\left[\eta,\Pi_{AG},\Theta;X_1,\cdots,X_N\right]$ is obtained by using the $\hat{\alpha}_{11,i},\hat{A}_{1,i},\hat{\alpha}_{2,i},\hat{A}_{22,i},\hat{\eta}_i,\hat{\Theta}_i$ and $\hat{\Omega}_{E,i}$ with i being chosen in a way that convergence criterion succeeded. The maximized likelihood function under the alternative hypothesis $H_A:\Pi_A(h)=\alpha A'$, $\max\limits_{H_A(h)} L\left[\eta,\Pi_{AG},\Theta;X_1,\cdots,X_N\right]$ is obtained in Section 3.4.1, Equation (3.46). Based on the results provided in Johansen-Juselius (1990) for their H_6 hypothesis, the test statistic is asymptotically χ^2 distributed. Since the derivation of the degrees of freedom is based on the number of free parameters in $\alpha A'$, it is equal to $kh - k_1h_1 - k_2h_2 - h_1h_2 + k_1k_2(P-1)$ as shown in Mosconi and Giannini (1992, p.416).

4.3.1 A Simulation Study

After temporal aggregation, the structure of the error correction model has a different form. The ECM-based non-causality tests in cointegrated systems should not be used without considering this fact. To see the effect of temporal aggregation on this test, we perform a simulation study. We choose a two-dimensional cointegrated VAR(1) model for the basic series with parameters $\boldsymbol{\phi}=\begin{bmatrix}1.0 & 0.0\\0.4 & 0.0\end{bmatrix}$ and $\Omega=\begin{bmatrix}1.0 & 0.5\\0.5 & 1.0\end{bmatrix}$. This model is chosen because the system is cointegrated. Although the first variable causes the second one, the second variable does not cause the first one. So, we can see the test results for both situations.

Before the simulation study, let's look at the effect of aggregation on the causality conditions. Consider the following 2-dimensional cointegrated VAR(1) process:

$$(\boldsymbol{I}-\boldsymbol{\phi}\mathrm{B})\boldsymbol{x}_t=\begin{bmatrix}1-\mathrm{B} & 0\\-0.4\mathrm{B} & 1\end{bmatrix}\begin{bmatrix}x_{1t}\\x_{2t}\end{bmatrix}=\boldsymbol{a}_t, \tag{4.71}$$

where $\boldsymbol{a}_t$ is white noise with mean vector $\mathbf{0}$ and the covariance matrix $\Omega=\begin{bmatrix}1.0 & 0.5\\0.5 & 1.0\end{bmatrix}$.

Then, the Wold representation of (4.71) is given by

$$(1\text{-B})\boldsymbol{x}_t=(\boldsymbol{I}+\Psi_1\mathrm{B})\boldsymbol{a}_t,$$

where $\Psi_1=\begin{bmatrix}\Psi_{11} & \Psi_{12}\\\Psi_{21} & \Psi_{22}\end{bmatrix}=\begin{bmatrix}0 & 0\\0.4 & -1\end{bmatrix}$. By Proposition 2.3, the aggregate series has the following form

$$(1-\mathrm{B})\boldsymbol{X}_T=(\boldsymbol{I}-\Theta_1\mathrm{B})\boldsymbol{E}_T, \tag{4.72}$$

where $\boldsymbol{E}_T$ is a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix Ω_E . For $m = 3$, we obtain the moving average parameter matrix as

$\Theta_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} -0.189 & -0.074 \\ -0.476 & 0.971 \end{bmatrix}$ and covariance matrix of aggregates is

$\Omega_E = \begin{pmatrix} 18.09 & 7.84 \\ 7.84 & 5.66 \end{pmatrix}$. By Lütkepohl (1991), the x_{2t} does not cause x_{1t} because

$\Psi_{12} = 0$. However, for the aggregates $\Theta_{12} = -0.074$, therefore X_{2T} does cause X_{1T}. Now, lets look at results of our modified test for non-causality in cointegrated systems using aggregates in order to see how well our test perform.

First, we generate a sample of size 600 and obtain its m^{th} order aggregates for various *m*. We then apply our modified test for non-causality with cointegration to this data set. The results are shown in Table 4.2. The results indicate that the modified statistic works well. Temporal aggregation indeed changes non-causal relationship into a causal one even when there is cointegration in the system.

4.4 An Empirical Example

As a real life example, the U.S. monthly wage and salary disbursements (WAGE) and real personal consumption expenditures (PCE) are selected from January 1976 to December 2000. The sources of the data are the Board of Governors of the Federal Reserve System and U.S. Department of Commerce: Bureau of Economic Analysis. The data are shown in Figure 4.1 with 300 observations. Both series have an increasing trend. The unit root test given in Table 4.3 also confirms this result.

Table 4.2 Modified Test Statistic for Non-Causality with Cointegration and Its p-Value for Various Aggregation Period

The first variable does not cause the second variable		
m	Chi-square	p-value
1	342.978	0
3	314.627	0
4	245.062	0
6	143.499	0
8	100.109	0
10	72.111	0
12	57.462	0
The second variable does not cause the first variable		
m	Chi-square	p-value
1	0.00966	09217
3	123.730	0
4	129.580	0
6	107.397	0
8	82.034	0
10	65.445	0
12	54.739	0

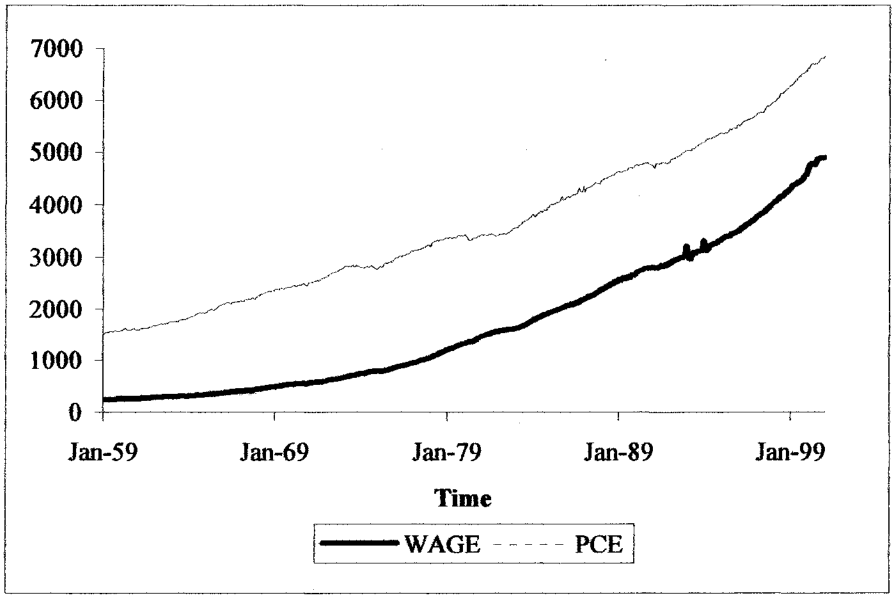


Figure 4.1 Time Series Plot of Monthly U.S. Wage and Salary Disbursements and Personal Consumption Expenditures

Table 4.3 The Dickey-Fuller Unit Root Test of U.S. Monthly WAGE and PCE

Variable	Type	Tau	Prob<Tau
WAGE	Zero Mean	11.56	0.9999
PCE	Zero Mean	11.65	0.9999

To be able to decide the order of the VARMA parameters, we looked at partial cross correlations of differenced series which are given in Table 4.4. This table indicates that data are generated possibly from VAR(2) process.

Table 4.4 Sample Partial Cross Correlations of U.S. Monthly WAGE and PCE

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
WAGE	+.	+-	-.	..	..	..	..	..	..	..	..	..
PCE	.+	-+	..	..	..	..	..	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

We perform a cointegration test for this series and the results given in Table 4.5. It implies that there is cointegration in the system with cointegration rank 1 at a 5% significance level.

Table 4.5 The Trace Test for Cointegration of U.S. Monthly WAGE and PCE Using Adjusted Test Statistic

H_0: Rank=h	H_A: Rank>h	Eigenvalue	Trace	Critical Value	Drift InECM	DriftIn Process
0	0	0.4033	154.07	12.21	NOINT	Constant
1	1	0.0006	0.18	4.14		

By using the results of Mosconi and Giannini (1992), the Granger non-causality test is applied. The results are given in Table 4.6. The degrees of freedom of the χ^2 test is 1. At a 5% level of significance, the null hypotheses of non-causality, stating the real personal consumption expenditures does not cause the wage and salary disbursements, is rejected whereas there is not enough evidence to reject the null hypothesis, the wage and salary disbursements does not cause the real personal consumption expenditures.

Table 4.7 Granger Non-Causality Test of U.S. Monthly WAGE and PCE

	χ^2 Test Statistics	p - Value
H_0: WAGE does not cause PCE	1.472	0.2249
H_0: PCE does not cause WAGE	24.228	1.7285E-6

Now we will analyze the temporally aggregated data. We use an aggregation period of $m = 3$ so that we have quarterly data with a sample size of 100. The time series plot is given in Figure 4.2. The non-stationary behavior of the variables can also be seen for quarterly data.

The MINIC in Table 4.7 suggests a VARMA(2,1) model while the schematic representation of partial cross correlations of differenced series given in Table 4.8 suggests a vector AR(2) model. Because of aggregate data, we will over fit and consider a VARMA(2,1) model in our analysis.

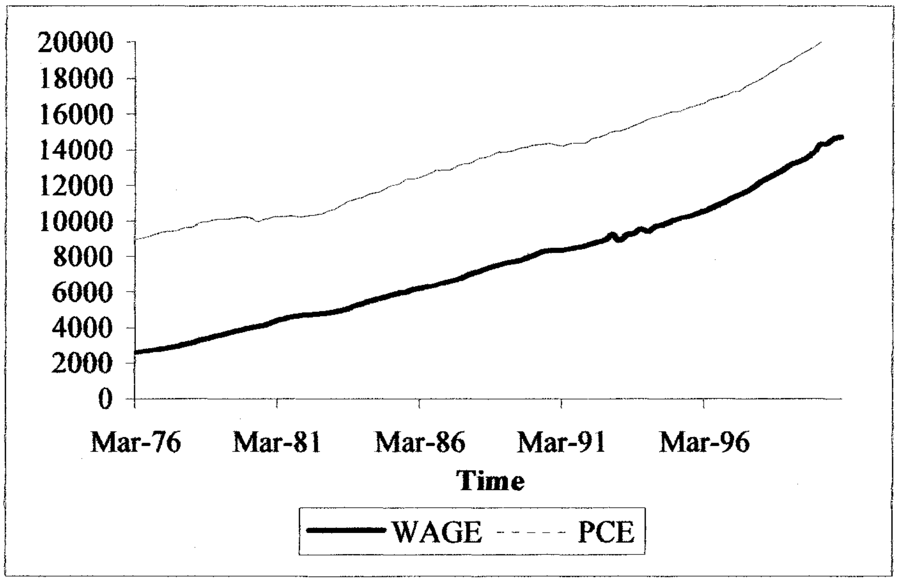


Figure 4.2 Time Series Plot of U.S. Quarterly WAGE and PCE

Table 4.7 Minimum Information Criterion for U.S. Quarterly WAGE and PCE

Lag	MA 0	MA 1	MA 2	MA 3	MA 4	MA 5
AR 0	33.548	33.691	33.732	33.771	33.823	33.874
AR 1	17.473	17.565	17.600	17.564	17.578	17.551
AR 2	17.406	17.392	17.546	17.570	17.610	17.637
AR 3	17.428	17.535	17.571	17.632	17.692	17.733
AR 4	17.431	17.525	17.611	17.704	17.775	17.836
AR 5	17.436	17.529	17.632	17.738	17.828	17.955

Table 4.8 Sample Partial Cross Correlations of U.S. Quarterly WAGE and PCE

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
WAGE	+.	-+	-.	..	..	..	..	..	..	..	..	..
PCE	++	-+	-.	..	..	..	..	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

To be able to conduct a cointegration test for aggregates, we must first estimate the parameters of vector ARMA (2, 1) for aggregates. Since we need the error terms in order to calculate the test statistic, we fit the differenced series by maximum likelihood estimation method for the model

$$\left(\boldsymbol{I}-\boldsymbol{\Phi}_1\mathrm{B}-\boldsymbol{\Phi}_2\boldsymbol{B}^2\right)\Delta\boldsymbol{X}_T=\left(\boldsymbol{I}-\boldsymbol{\Theta}\mathrm{B}\right)\boldsymbol{E}_T.$$

The MA representation of this model can be found by

$$\Delta\boldsymbol{X}_T=\left(\boldsymbol{I}-\boldsymbol{\Phi}_1\mathrm{B}-\boldsymbol{\Phi}_2\boldsymbol{B}^2\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{\Theta}\mathrm{B}\right)\boldsymbol{E}_T.$$

The ECM is given by

$$\Delta\boldsymbol{X}_T=\eta\Delta\boldsymbol{X}_{T-1}+\Pi_{AG}\boldsymbol{X}_{T-2}+\boldsymbol{E}_T-\boldsymbol{\Theta}\boldsymbol{E}_{T-1},$$

where $\boldsymbol{X}_T=\left(\mathrm{X}_{1T}\quad \mathrm{X}_{2T}\right)'$, $\boldsymbol{E}_T=\left(\mathrm{E}_{1T}\quad \mathrm{E}_{2T}\right)'$ and $\Pi_{AG}=\boldsymbol{\alpha}\boldsymbol{A}'$.

Since the error terms in VARMA model and ECM are the same, after maximum likelihood estimation of the parameters of VARMA model, we can obtain the residuals, $\hat{\boldsymbol{E}}_{T-1}$ and use them to calculate the test statistic. The results given in Table 4.9 imply that the aggregates are also cointegrated with rank 1 at 5% significance level.

Table 4.9 The Trace Test for Cointegration of U.S. Quarterly WAGE and PCE Using Adjusted Test Statistic

H_0: Rank=h	H_A: Rank>h	Eigenvalue	Trace	Critical Value	Drift InECM	DriftIn Process
0	0	0.1768	32.53	12.21	NOINT	Constant
1	1	0.0000	0.41	4.14		

At last we test whether there is a causal relationship between aggregated variables in the cointegrated system by using the new testing approach developed in this chapter. Table 4.10 shows the results of a modified non-causality test with cointegration for aggregates. Clearly, there is a causal relationship between variables in both ways at a 5% level of significance.

Table 4.10 The Granger Non-Causality Test of U.S. Quarterly WAGE and PCE Using Adjusted Test Statistic

	χ^2 value	p - value
H_0: WAGE does not cause PCE	28.2956	1.192E-7
H_0: PCE does not cause WAGE	30.3339	5.961E-8

CHAPTER 5

REPRESENTATION OF MULTIPLICATIVE VECTOR AUTOREGRESSIVE MOVING AVERAGE PROCESSES

5.1 Introduction

Many time series data have a seasonal behavior. In order to analyze them, multiplicative time series models are generally used. In this chapter, we will consider different representations of multiplicative processes and provide a guideline to be used to select the best representation.

Let $\{x_t\}$, $t = 0, \pm 1, \pm 2, \ldots$ be a univariate zero mean, covariance stationary, purely non-deterministic, multiplicative autoregressive and moving average model with seasonal period s

$$\phi_p(B)\Phi_P(B^s)x_t = \theta_q(B)\Theta_Q(B^s)a_t, \tag{5.1}$$

where B is the back shift operator, $Bx_t = x_{t-1}$,

$$\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \cdots - \Phi_P B^{Ps},$$

$$\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q,$$

$$\Theta_Q(B^s) = 1 - \Theta_1 B^s - \cdots - \Theta_Q B^{Qs},$$

and a_t is the Gaussian white noise process with mean 0 and a constant variance σ_a^2. The model is often denoted as $ARMA(p,q)\times(P,Q)_s$. For our study, we will denote it as $ARMA(p)(P)_s(q)(Q)_s$. When $p = 0$ and $P = 0$, the model is referred to as a multiplicative moving average model of order q and Q with a seasonal period s, and it is shortened as $MA(q)(Q)_s$. When $q = 0$ and $Q = 0$, the model is referred to as a multiplicative autoregressive model of order p and P with a seasonal period s, and it is shortened as $AR(p)(P)_s$. When $\boldsymbol{x}_t$ is a k-dimensional vector, the natural extension is the following multiplicative vector autoregressive and moving average $VARMA(p)(P)_s(q)(Q)_s$ model,

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{\Phi}_P(\mathrm{B}^s)\underset{k\times 1}{\boldsymbol{x}_t} = \boldsymbol{\theta}_q(\mathrm{B})\boldsymbol{\Theta}_Q(\mathrm{B}^s)\underset{k\times 1}{\boldsymbol{a}_t}, \tag{5.2}$$

where

$$\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\phi}_1\mathrm{B} - \ldots - \boldsymbol{\phi}_p\mathrm{B}^p,$$

$$\boldsymbol{\Phi}_P(\mathrm{B}^s) = \boldsymbol{I}_k - \boldsymbol{\Phi}_1\mathrm{B}^s - \ldots - \boldsymbol{\Phi}_P\mathrm{B}^{sP},$$

$$\boldsymbol{\theta}_q(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\theta}_1\mathrm{B} - \ldots - \boldsymbol{\theta}_q\mathrm{B}^q,$$

and

$$\boldsymbol{\Theta}_Q(\mathrm{B}^s) = \boldsymbol{I}_k - \boldsymbol{\Theta}_1\mathrm{B}^s - \ldots - \boldsymbol{\Theta}_Q\mathrm{B}^{sQ},$$

are matrix polynomials in the backshift operator B, defined by $\mathrm{B}^j\boldsymbol{V}_t = \boldsymbol{V}_{t-j}$ for any integer j and vector $\boldsymbol{V}_t$. $\boldsymbol{I}_k$ is the k-dimensional identity matrix, the $\boldsymbol{\phi}$'s, $\boldsymbol{\Phi}$'s, $\boldsymbol{\theta}$'s and $\boldsymbol{\Theta}$'s are $k \times k$ parameter matrices, and the $\boldsymbol{a}_t$ vector Gaussian white noise process with mean vector $\boldsymbol{0}$ and $\mathrm{E}(\boldsymbol{a}_t\boldsymbol{a}_t') = \boldsymbol{\Omega}$. When $p = 0$ and $P = 0$, the model is referred to as a multiplicative vector moving average model of order q and Q with a seasonal period s, and it is shortened to $VMA(q)(Q)_s$. When $q = 0$ and $Q = 0$, the model is referred to as a multiplicative vector autoregressive model of order p and P with a seasonal period s, and it is shortened to $VAR(p)(P)_s$.

Because of the commutative nature of scalars in a univariate time series, the $ARMA(p)(P)_s(q)(Q)_s$ model in (5.1) can also be written as the following $ARMA(P)_s(p)(Q)_s(q)$ model

$$\Phi_P(B^s)\phi_p(B)x_t = \Theta_Q(B^s)\theta_q(B)a_t. \tag{5.3}$$

Can we extend this operation to the vector process and write the $VARMA(p)(P)_s(q)(Q)_s$ model in (5.2) as the following $VARMA(P)_s(p)(Q)_s(q)$ model?

$$\boldsymbol{\Phi}_P(\mathrm{B}^s)\boldsymbol{\phi}_p(\mathrm{B})\underset{k\times 1}{\boldsymbol{x}_t} = \boldsymbol{\Theta}_Q(\mathrm{B}^s)\boldsymbol{\theta}_q(\mathrm{B})\underset{k\times 1}{\boldsymbol{a}_t}. \tag{5.4}$$

Because the matrix multiplication is non-commutative, the answer may likely be “no”. Both representations (5.2) and (5.4) have been used in the literature and yet surprisingly the problem has never been raised and discussed.

5.2 Representations and Estimation of a Multiplicative Vector Process

5.2.1 Maximum Likelihood Estimation of Multivariate VARMA Processes

In considering the multiplicative vector AR time series model with seasonal period s, we can represent it as $VAR(p)(P)_s$

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{\Phi}_P(\mathrm{B}^s)\underset{k\times 1}{\boldsymbol{x}_t} = \underset{k\times 1}{\boldsymbol{a}_t}, \tag{5.5}$$

or $VAR(P)_s(p)$

$$\boldsymbol{\Phi}_P(\mathrm{B}^s)\boldsymbol{\phi}_p(\mathrm{B})\underset{k\times 1}{\boldsymbol{x}_t} = \underset{k\times 1}{\boldsymbol{a}_t}, \tag{5.6}$$

where $\boldsymbol{a}_t$s are i.i.d. $N(\mathbf{0},\Omega)$ and B is the backshift operator such that $\mathrm{B}\boldsymbol{x}_t = \boldsymbol{x}_{t-1}$.

To see whether representations (5.5) and (5.6) lead to the same estimation, let us consider a simple $VAR(1)(1)_s$

$$(\boldsymbol{I}-\phi\mathrm{B})(\boldsymbol{I}-\Phi\mathrm{B}^s)\boldsymbol{x}_t=\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\phi\Phi\boldsymbol{x}_{t-s-1}=\boldsymbol{a}_t, \qquad (5.7)$$

where $\boldsymbol{a}_t$s are i.i.d. $\mathrm{N}(\mathbf{0},\boldsymbol{I})$. For simplicity, we assume that ϕ is known. The likelihood function for (5.7) can be written as

$$\ell_1(\Phi\,|\,\boldsymbol{x})\propto\exp\left\{-\frac{1}{2}\mathrm{tr}\sum_{t=1}^{n}\left(\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\phi\Phi\boldsymbol{x}_{t-s-1}\right)\left(\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\phi\Phi\boldsymbol{x}_{t-s-1}\right)'\right\}. \qquad (5.8)$$

For a fixed ϕ, the maximum likelihood estimator of Φ can be found from

$$\Phi\sum_{t=1}^{n}\boldsymbol{x}_{t-s}\boldsymbol{x}_{t-s}'-\phi'\Phi\sum_{t=1}^{n}\boldsymbol{x}_{t-s}\boldsymbol{x}_{t-s-1}'-\phi\Phi\sum_{t=1}^{n}\boldsymbol{x}_{t-s-1}\boldsymbol{x}_{t-s}'+\phi'\phi\Phi\sum_{t=1}^{n}\boldsymbol{x}_{t-s-1}\boldsymbol{x}_{t-s-1}'= \\ =\sum_{t=1}^{n}\left(\boldsymbol{x}_t\boldsymbol{x}_{t-s}'-\phi\boldsymbol{x}_t\boldsymbol{x}_{t-s-1}'-\phi\boldsymbol{x}_{t-1}\boldsymbol{x}_{t-s}'+\phi\boldsymbol{x}_{t-1}\boldsymbol{x}_{t-s-1}'\phi'\right). \qquad (5.9)$$

Consider the following $VAR(1)_s(1)$ representation

$$(\boldsymbol{I}-\Phi\mathrm{B}^s)(\boldsymbol{I}-\phi\mathrm{B})\boldsymbol{x}_t=\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\Phi\phi\boldsymbol{x}_{t-s-1}=\boldsymbol{a}_t, \qquad (5.10)$$

where ϕ is known and $\boldsymbol{a}_t$s are i.i.d. $\mathrm{N}(\mathbf{0},\boldsymbol{I})$. The likelihood function of (5.10) can be written as

$$\ell_2(\Phi\,|\,\boldsymbol{x})\propto\exp\left\{-\frac{1}{2}\mathrm{tr}\sum_{t=1}^{n}\left(\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\Phi\phi\boldsymbol{x}_{t-s-1}\right)\left(\boldsymbol{x}_t-\phi\boldsymbol{x}_{t-1}-\Phi\boldsymbol{x}_{t-s}+\Phi\phi\boldsymbol{x}_{t-s-1}\right)'\right\}. \qquad (5.11)$$

The maximum likelihood estimator of Φ is

$$\hat{\Phi}=\left[\sum_{t=1}^{n}\left(x_t x'_{t-s}-\phi x_{t-1}x'_{t-s}-x_t x'_{t-s-1}\phi'+\phi x_{t-1}x'_{t-s-1}\phi\right)\right]\times$$
$$\times\left[\sum_{t=1}^{n} x_{t-s}x'_{t-s}-\phi x_{t-s-1}x'_{t-s}-x_{t-s}x'_{t-s-1}\phi'+\phi x_{t-s-1}x'_{t-s-1}\phi'\right]^{-1} \quad (5.12)$$

From (5.9) and (5.12) we can see that the maximum likelihood estimators of the same parameter are different for the two representations. The aim of the following simulation study is to illustrate the differences in estimation for each representation type.

5.2.2 Simulation Study

If we have a multiplicative VAR process, we can represent it by the following processes

$$\phi_p(\mathrm{B})\Phi_P(\mathrm{B}^s)x_t=a_t, \quad (5.13)$$

or

$$\Phi_P(\mathrm{B}^s)\phi_p(\mathrm{B})x_t=a_t, \quad (5.14)$$

where a_ts are i.i.d. $N(\mathbf{0},\Omega)$, and B is the backshift operator such that $\mathrm{B}x_t=x_{t-1}$. The purpose of the following simulation study is to show the differences in estimation and forecasting for these representations.

To investigate the impact of different representations on parameter estimation, 1,105 observations were generated from the following two-dimensional vector model

$$(I-\phi\mathrm{B})(I-\Phi\mathrm{B}^4)x_t=a_t, \quad (5.15)$$

with $\phi=\begin{bmatrix}.5 & -.4\\ .3 & .6\end{bmatrix}$, $\Phi=\begin{bmatrix}.6 & .1\\ -.6 & .8\end{bmatrix}$, and a_t is the vector Gaussian white noise

$N\left(\begin{bmatrix}0\\0\end{bmatrix}, \Omega = \begin{bmatrix}2 & .5\\ .5 & 1.5\end{bmatrix}\right)$. To remove the effect of initial values, the first 100 observations were deleted, the last 5 observations were retained for forecasting comparison used in the next section, and the remaining 1,000 observations were used to estimate the parameters with two different case representations:

CASE 1: Fit the series as $VAR(1)(1)_4$,

CASE 2: Fit the series as $VAR(1)_4(1)$.

This process was repeated 10,000 times.

Similarly, 1,105 observations were generated from the two-dimensional vector model

$$(\boldsymbol{I} - \Phi B^4)(\boldsymbol{I} - \phi B)\boldsymbol{x}_t = \boldsymbol{a}_t, \qquad (5.16)$$

with the same ϕ, Φ, and $\boldsymbol{a}_t$ given in (5.15), and the above process was repeated to estimate the parameters in the following case representations:

CASE 3: Fit the series as $VAR(1)(1)_4$,

CASE 4: Fit the series as $VAR(1)_4(1)$.

Again, the process was repeated 10,000 times.

Table 5.1 gives averages of the estimates for ϕ, Φ, and Ω for Case 1 and Case 4 clearly give much smaller biases than for Case 2 and Case 3. The sum of absolute biases for all parameters for Case 1 is 0.019, whereas for Case 2 it is 0.776. Similarly, the values for Case 3 and Case 4 are 1.282 and 0.02, respectively. The results imply that the correct representation of a multiplicative vector model is important. The $VARMA(p)(P)_s(q)(Q)_s$ and $VARMA(P)_s(p)(Q)_s(q)$ representations are not interchangeable. Similar results were obtained for the VMA and VARMA processes.

Table 5.1 Average Maximum Likelihood Estimates of Parameters of Various Representations

PARAMETERS									
ϕ		CASE 1		CASE 2		CASE 3		CASE 4	
0.5	**-0.4**	0.499	-0.401	0.585	-0.157	0.521	-0.377	0.499	-0.4
0.3	**0.6**	0.3	0.599	0.312	0.591	0.332	0.626	0.3	0.599
Φ									
0.6	**0.1**	0.599	0.101	0.59	0.096	0.741	0.156	0.598	0.101
-0.6	**0.8**	-0.6	0.799	-0.607	0.792	-0.38	0.822	-0.601	0.799
Ω									
2	**0.5**	1.994	0.499	2.246	0.505	2.223	0.635	1.996	0.499
0.5	**1.5**	0.499	1.494	0.505	1.552	0.635	1.748	0.499	1.493

5.3 Forecasting

One of the most important applications of the time series analysis is to forecast or predict future values. The predictor that minimizes the forecast mean squared errors (MSEs) is the most widely used.

Given $t = 1, 2, ..., n$, the minimum mean square error forecast of $\boldsymbol{x}_{n+\ell}$ is given by its conditional expectation $\hat{\boldsymbol{x}}_n(\ell)$. That is,

$$\hat{\boldsymbol{x}}_n(\ell) = \mathrm{E}(\boldsymbol{x}_{n+\ell} \mid \boldsymbol{x}_n, \boldsymbol{x}_{n-1}, \ldots), \tag{5.17}$$

which is called the ℓ-step ahead forecast of $\boldsymbol{x}_{n+\ell}$ at the forecast origin n where $\ell = 1,2,\ldots$ The corresponding vector of forecast errors is

$$\boldsymbol{e}_n(\ell) = \boldsymbol{x}_{n+\ell} - \hat{\boldsymbol{x}}_n(\ell). \tag{5.18}$$

Forecast comparisons are often based on Mean Square Forecast Error (MSFE) defined for a univariate process as

$$\mathrm{E}[\boldsymbol{e}_n^2(\ell)] = \{\mathrm{E}[\boldsymbol{e}_n(\ell)]\}^2 + \mathrm{V}[\boldsymbol{e}_n(\ell)], \tag{5.19}$$

which is equal to the squared bias in the ℓ-step ahead forecast error $\boldsymbol{e}_n(\ell) = \boldsymbol{x}_{n+\ell} - \hat{\boldsymbol{x}}_n(\ell)$ plus the forecast error variance.

In a multivariate analysis, the MSFE matrix is $\mathrm{E}[\boldsymbol{e}_n(\ell)\boldsymbol{e}_n'(\ell)]$, but usually the trace MSFE (TMSFE) is used to compare forecasts (Lin and Tsay, 1996). Thus, TMSFE is the sum of the MSFE for each variable and denoted by $\mathrm{tr}\{\mathrm{E}[\boldsymbol{e}_n(\ell)\boldsymbol{e}_n'(\ell)]\}$. The sample MSFE is calculated by

$$\hat{\delta}(\ell) = \boldsymbol{e}_n(\ell)\boldsymbol{e}_n'(\ell). \tag{5.20}$$

In forecasting comparison through a simulation, after the TMSFEs are obtained, the square root average for ℓ-step ahead forecasts is computed as

$$\boldsymbol{M}(\ell) = \sqrt{\sum_{i=1}^{R} \mathrm{tr}(\hat{\delta}_i(\ell))/\mathrm{R}}\,, \tag{5.21}$$

where the summation is over the number of replications (R = 10,000). Table 5.2 shows the square root averages of TMSFEs for the various ℓ-step ahead forecasts from the models generated earlier in section 5.2.2 for the two representations in (5.15) and (5.16), including their estimation results.

From this table we can see that the square root averages of the TMSFE for Case 1 and Case 4 are clearly less than those for Case 2 and Case 3. This implies that the correct representation of a multiplicative model is important to achieve good forecast results.

Table 5.2 Five-step Ahead $M(\ell)$ Values of Various Representations

	ℓ=1	ℓ=2	ℓ=3	ℓ=4	ℓ=5
CASE 1	1.875	2.228	2.362	2.428	2.676
CASE 2	1.957	2.342	2.462	2.506	2.803
CASE 3	2.001	2.437	2.618	2.699	2.994
CASE 4	1.875	2.230	2.366	2.430	2.678

5.4 Representations and Causality

One of the most important goals in studying vector time series is to examine the causal relationship between variables. There are many measures for causal relationships between variables; the Granger (1969) causality is probably the most widely used. Let the k dimensional vector process be partitioned into two subvectors $\boldsymbol{x}_t = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t})'$ where $\boldsymbol{x}_{1t} = (\mathrm{y}_{1t}, \ldots, \mathrm{y}_{k_1 t})'$ and $\boldsymbol{x}_{2t} = (\mathrm{y}_{k_1+1}, \ldots, \mathrm{y}_{k_1+k_2})'$ are k_1 and k_2 dimensional vectors respectively, and $k = k_1 + k_2$. A time series $\{\boldsymbol{x}_{1t}\}$ is said to cause another time series $\{\boldsymbol{x}_{2t}\}$ if the present value of $\boldsymbol{x}_2$ can be better predicted by using the past values of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ rather than by using only the past values of $\boldsymbol{x}_2$.

Consider the stationary and invertible k-dimensional VARMA (p, q) process

$$\Pi(\boldsymbol{B})\boldsymbol{x}_t = \varphi(\boldsymbol{B})\boldsymbol{a}_t, \qquad (5.22)$$

where $\Pi(\mathrm{B}) = \boldsymbol{I} - \Pi_1\mathrm{B} - \ldots - \Pi_p\mathrm{B}^p = \begin{pmatrix} \Pi_{11}(\mathrm{B}) & \Pi_{12}(\mathrm{B}) \\ \Pi_{21}(\mathrm{B}) & \Pi_{22}(\mathrm{B}) \end{pmatrix}$ and $\varphi(\mathrm{B}) = \boldsymbol{I} - \varphi_1\mathrm{B} - \ldots - \varphi_q\mathrm{B}^q = \begin{pmatrix} \varphi_{11}(\mathrm{B}) & \varphi_{12}(\mathrm{B}) \\ \varphi_{21}(\mathrm{B}) & \varphi_{22}(\mathrm{B}) \end{pmatrix}$. The $\boldsymbol{a}_t$'s are uncorrelated random vectors with mean $\mathbf{0}$ and the non-singular covariance matrix $\boldsymbol{\Omega}$. Assume that the parameters in $\Pi(\mathrm{B})$ and $\varphi(\mathrm{B})$ are uniquely defined and the process $\boldsymbol{x}_t$ is partitioned into two vectors $\boldsymbol{x}_t = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t})'$, where $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ are k_1 and k_2 dimensional vectors respectively with $k_1 + k_2 = k$. Then, $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if

$$\Pi_{21}(z) - \varphi_{21}(z)\varphi_{11}(z)^{-1}\Pi_{11}(z) = 0, \tag{5.23}$$

(Boudjellaba, Dufour and Roy, 1992).

In the following sections, we will examine the causality conditions for two different representations for some simple multiplicative vector processes.

5.4.1 Vector Autoregressive Process

Consider the two-dimensional multiplicative vector autoregressive model with p = 1, P = 1, and a seasonal period s. For this process, we have two possible representations: $VAR(1)(1)_s$,

$$(\boldsymbol{I} - \phi\mathrm{B})(\boldsymbol{I} - \Phi\mathrm{B}^s)\boldsymbol{x}_t = \boldsymbol{a}_t, \tag{5.24}$$

which is more explicitly written as

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}\mathrm{B}\right]\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}\mathrm{B}^s\right]\boldsymbol{x}_t = \boldsymbol{a}_t,$$

$$\begin{bmatrix} 1 - \phi_{11}\mathrm{B} - \Phi_{11}\mathrm{B}^s + (\phi_{11}\Phi_{11} + \phi_{12}\Phi_{21})\mathrm{B}^{s+1} & -\phi_{12}\mathrm{B} - \Phi_{12}\mathrm{B}^s + (\phi_{11}\Phi_{12} + \phi_{12}\Phi_{22})\mathrm{B}^{s+1} \\ -\phi_{21}\mathrm{B} - \Phi_{21}\mathrm{B}^s + (\phi_{21}\Phi_{11} + \phi_{22}\Phi_{21})\mathrm{B}^{s+1} & 1 - \phi_{22}\mathrm{B} - \Phi_{22}\mathrm{B}^s + (\phi_{21}\Phi_{12} + \phi_{22}\Phi_{22})\mathrm{B}^{s+1} \end{bmatrix}\boldsymbol{x}_t = \boldsymbol{a}_t,$$

and $VAR(1)_s(1)$,

$$(\boldsymbol{I} - \Phi\mathrm{B}^s)(\boldsymbol{I} - \phi\mathrm{B})\boldsymbol{x}_t = \boldsymbol{a}_t, \tag{5.25}$$

or more explicitly

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{pmatrix}\mathrm{B}^s\right]\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}\mathrm{B}\right]\boldsymbol{x}_t=\boldsymbol{a}_t,$$

$$\begin{bmatrix}1-\phi_{11}\mathrm{B}-\Phi_{11}\mathrm{B}^s+(\Phi_{11}\phi_{11}+\Phi_{12}\phi_{21})\mathrm{B}^{s+1} & -\phi_{12}\mathrm{B}-\Phi_{12}\mathrm{B}^s+(\Phi_{11}\phi_{12}+\Phi_{12}\phi_{22})\mathrm{B}^{s+1}\\ -\phi_{21}\mathrm{B}-\Phi_{21}\mathrm{B}^s+(\Phi_{21}\phi_{11}+\Phi_{22}\phi_{21})\mathrm{B}^{s+1} & 1-\phi_{22}\mathrm{B}-\Phi_{22}\mathrm{B}^s+(\Phi_{21}\phi_{12}+\Phi_{22}\phi_{22})\mathrm{B}^{s+1}\end{bmatrix}\boldsymbol{x}_t=\boldsymbol{a}_t.$$

According to (5.23), the $VAR(1)(1)_s$ representation given in (5.24), $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\phi_{21}=0$, $\Phi_{21}=0$ and $\phi_{21}\Phi_{11}+\phi_{22}\Phi_{21}=0$. This means that $\phi_{21}=0$ and $\Phi_{21}=0$ are the non-causality conditions for (5.24) because in Equation (5.23)

$$\Pi_{11}(z)=1-\phi_{11}z-\Phi_{11}z^s+(\phi_{11}\Phi_{11}+\phi_{12}\Phi_{21})z^{s+1},\ \Pi_{21}(z)=-\phi_{21}z-\Phi_{21}z^s+(\phi_{21}\Phi_{11}+\phi_{22}\Phi_{21})z^{s+1},$$

$$\varphi_{11}(z)=1 \text{ and } \varphi_{21}(z)=0.$$

According to (5.23), for the $VAR(1)_s(1)$ representation given in (5.25), $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\phi_{21}=0$, $\Phi_{21}=0$, and $\phi_{11}\Phi_{21}+\phi_{21}\Phi_{22}=0$. This implies that $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\phi_{21}=0$ and $\Phi_{21}=0$ since in Equation (5.23)

$$\Pi_{11}(z)=1-\phi_{11}z-\Phi_{11}z^s+(\Phi_{11}\phi_{11}+\Phi_{12}\phi_{21})z^{s+1},\ \Pi_{21}(z)=-\phi_{21}z-\Phi_{21}z^s+(\Phi_{21}\phi_{11}+\Phi_{22}\phi_{21})z^{s+1},$$

$$\varphi_{11}(z)=1 \text{ and } \varphi_{21}(z)=0.$$

Thus, in both representations, we have the same non-causality conditions.

5.4.2 Vector Autoregressive-Moving Average Process

Consider the simple multiplicative VARMA process with $p = 1$, $q = 1$, $P = 1$ and $Q = 1$. For this process, let us consider the following two different representations:

$$(\boldsymbol{I}-\boldsymbol{\phi}\mathrm{B})(\boldsymbol{I}-\boldsymbol{\Phi}\mathrm{B}^s)\boldsymbol{x}_t=(\boldsymbol{I}-\boldsymbol{\theta}\mathrm{B})(\boldsymbol{I}-\boldsymbol{\Theta}\mathrm{B}^s)\boldsymbol{a}_t, \tag{5.26}$$

which is

$$\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\phi_{11}&\phi_{12}\\\phi_{21}&\phi_{22}\end{pmatrix}\mathrm{B}\right]\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\Phi_{11}&\Phi_{12}\\\Phi_{21}&\Phi_{22}\end{pmatrix}\mathrm{B}^s\right]\boldsymbol{x}_t =$$

$$=\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\theta_{11}&\theta_{12}\\\theta_{21}&\theta_{22}\end{pmatrix}\mathrm{B}\right]\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\Theta_{11}&\Theta_{12}\\\Theta_{21}&\Theta_{22}\end{pmatrix}\mathrm{B}^s\right]\boldsymbol{a}_t,$$

and

$$(\boldsymbol{I}-\Phi\mathrm{B}^s)(\boldsymbol{I}-\phi\mathrm{B})\boldsymbol{x}_t=(\boldsymbol{I}-\Theta\mathrm{B}^s)(\boldsymbol{I}-\theta\mathrm{B})\boldsymbol{a}_t, \tag{5.27}$$

which is

$$\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\Phi_{11}&\Phi_{12}\\\Phi_{21}&\Phi_{22}\end{pmatrix}\mathrm{B}^s\right]\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\phi_{11}&\phi_{12}\\\phi_{21}&\phi_{22}\end{pmatrix}\mathrm{B}\right]\boldsymbol{x}_t =$$

$$=\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\Theta_{11}&\Theta_{12}\\\Theta_{21}&\Theta_{22}\end{pmatrix}\mathrm{B}^s\right]\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\theta_{11}&\theta_{12}\\\theta_{21}&\theta_{22}\end{pmatrix}\mathrm{B}\right]\boldsymbol{a}_t.$$

For the representation in (5.26), in using (5.23), $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\theta_{21}=\phi_{21}$, $\phi_{11}=\theta_{11}$, $\phi_{12}=\theta_{12}$, $\phi_{22}=\theta_{22}$, $\Phi_{21}=\Theta_{21}$ and $\Phi_{11}=\Theta_{11}$ because in Equation (5.23)

$$\Pi_{11}(z)=1-\phi_{11}z-\Phi_{11}z^s+(\phi_{11}\Phi_{11}+\phi_{12}\Phi_{21})z^{s+1},$$

$$\Pi_{21}(z)=-\phi_{21}z-\Phi_{21}z^s+(\phi_{21}\Phi_{11}+\phi_{22}\Phi_{21})z^{s+1},$$

and

$$\varphi_{11}(z)=1-\theta_{11}z-\Theta_{11}z^s+(\theta_{11}\Theta_{11}+\theta_{12}\Theta_{21})z^{s+1},$$

$$\varphi_{21}(z)=-\theta_{21}z-\Theta_{21}z^s+(\theta_{21}\Theta_{11}+\theta_{22}\Theta_{21})z^{s+1}.$$

For the representation in (5.27), $\boldsymbol{x}_1$ does not cause $\boldsymbol{x}_2$ if and only if $\theta_{21}=\phi_{21}$, $\phi_{11}=\theta_{11}$, $\Phi_{21}=\Theta_{21}$, $\Phi_{11}=\Theta_{11}$, $\Phi_{12}=\Theta_{12}$ and $\Phi_{22}=\Theta_{22}$ since in Equation (5.23)

$$\Pi_{11}(z)=1-\phi_{11}z-\Phi_{11}z^s+(\Phi_{11}\phi_{11}+\Phi_{12}\phi_{21})z^{s+1},$$

$$\Pi_{21}(z)=-\phi_{21}z-\Phi_{21}z^s+(\Phi_{21}\phi_{11}+\Phi_{22}\phi_{21})z^{s+1},$$

and

$$\varphi_{11}(z) = 1 - \theta_{11}z - \Theta_{11}z^s + (\Theta_{11}\theta_{11} + \Theta_{12}\theta_{21})z^{s+1},$$

$$\varphi_{21}(z) = -\theta_{21}z - \Theta_{21}z^s + (\Theta_{21}\theta_{11} + \Theta_{22}\theta_{21})z^{s+1}.$$

Thus, the causality conditions for the representation in (5.26) are $\theta_{21} = \phi_{21}$, $\phi_{11} = \theta_{11}$, $\phi_{12} = \theta_{12}$, $\phi_{22} = \theta_{22}$, $\Phi_{21} = \Theta_{21}$ and $\Phi_{11} = \Theta_{11}$, whereas the causality conditions for the representation in (5.27) are $\theta_{21} = \phi_{21}$, $\phi_{11} = \theta_{11}$, $\Phi_{21} = \Theta_{21}$, $\Phi_{11} = \Theta_{11}$, $\Phi_{12} = \Theta_{12}$ and $\Phi_{22} = \Theta_{22}$. They are no longer the same.

5.5 Summary Statistic to Determine the Multiplicative VARMA Model

The Akaike Information Criterion (AIC) (Akaike, 1973) is widely used for model selection. However, AIC is designed for minimizing the 1-step forecast mean square error and is not consistent (Shibata, 1980). Thus, the following consistent criteria are also taken into consideration; the Hannan and Quinn Information Criterion (HQ) (Hannan & Quinn, 1979), and the Schwarz Information Criteria (SC) (Schwarz, 1978). Our purpose of using the information criteria here is not to choose the order of the process but to decide the best representation for the multiplicative processes which have the same order. The following formulas for each criterion are used:

$$AIC = \ln|\hat{\Omega}| + \frac{2k^2(p+q+P+Q)}{n}, \tag{5.28}$$

$$HQ = \ln|\hat{\Omega}| + \frac{2k^2(p+q+P+Q)\ln(\ln n)}{n}, \tag{5.29}$$

$$SC = \ln|\hat{\Omega}| + \frac{k^2(p+q+P+Q)\ln n}{n}, \tag{5.30}$$

where n is the sample size, k is the dimension of the process, and $\hat{\Omega}$ is the maximum likelihood estimate of the error covariance matrix.

The equations (5.28), (5.29), and (5.30) were calculated for each case by using the same Monte Carlo results in section 5.2.2. Table 5.3 presents the averages of these values by using 10,000 realizations.

Table 5.3 Averages of Information Criteria of Various Representations

	AIC	HQ	SC
CASE 1	1.018	1.033	1.057
CASE 2	1.185	1.200	1.224
CASE 3	1.260	1.275	1.299
CASE 4	1.018	1.033	1.057

When we compare the information criteria of Case 2 with Case 1 and Case 3 with Case 4, we can see the slight superiority of Case 1 and Case 4 over Case 2 and Case 3 respectively because of the smaller values of AIC, HQ, and SC. All the information criteria give the same results so that one can use any of these values to decide the best multiplicative model representation. The smaller the information criterion is, the better the representation of the multiplicative process.

5.6 Model Structure of Temporal Aggregates in Multiplicative Seasonal Vector Time Series

For this section, we derive the proper model of temporal aggregates for a given basic seasonal vector time series process. Wei (1978b) showed that given a univariate time series model of the order $(p)(P)_s(q)(Q)_s$, the corresponding model for aggregates with an aggregation period m is of the order (p)(P)S(c)(Q)S, where $s = mS$ for some integer S and $c = \left[p + 1 + \frac{q-p-1}{m}\right]$. This means that after temporal aggregation, the order of the process is dependent only upon that of the moving average order of non-seasonal parameters. The following proposition is to generalize Wei's result to the multivariate case.

Proposition 5.1 Let $\boldsymbol{x}_t$ be a zero mean basic time series following a VARMA(p)(P)$_s$(q)(Q)$_s$ process:

$$\boldsymbol{\phi}_p(\mathrm{B})\boldsymbol{\Phi}_P(\mathrm{B}^s)\boldsymbol{x}_t = \boldsymbol{\theta}_q(\mathrm{B})\boldsymbol{\Theta}_Q(\mathrm{B}^s)\boldsymbol{a}_t, \tag{5.32}$$

where

$$\boldsymbol{\phi}_p(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\phi}_1\mathrm{B} - \ldots - \boldsymbol{\phi}_p\mathrm{B}^p,$$

$$\boldsymbol{\Phi}_P(\mathrm{B}^s) = \boldsymbol{I}_k - \boldsymbol{\Phi}_1\mathrm{B}^s - \ldots - \boldsymbol{\Phi}_P\mathrm{B}^{sP},$$

$$\boldsymbol{\theta}_q(\mathrm{B}) = \boldsymbol{I}_k - \boldsymbol{\theta}_1\mathrm{B} - \ldots - \boldsymbol{\theta}_q\mathrm{B}^q,$$

and

$$\boldsymbol{\Theta}_Q(\mathrm{B}^s) = \boldsymbol{I}_k - \boldsymbol{\Theta}_1\mathrm{B}^s - \ldots - \boldsymbol{\Theta}_Q\mathrm{B}^{sQ},$$

the $\boldsymbol{\phi}$s, $\boldsymbol{\Phi}$s, $\boldsymbol{\theta}$s, and $\boldsymbol{\Theta}$s are $k \times k$ parameter matrices, and $\boldsymbol{a}_t$ the vector Gaussian white noise process with mean vector $\mathbf{0}$ and $\mathrm{E}(\boldsymbol{a}_t\boldsymbol{a}_t') = \boldsymbol{\Omega}$. The aggregate time series defined by $\boldsymbol{X}_T = (1+\mathrm{B}+\ldots+\mathrm{B}^{m-1})\boldsymbol{x}_{mT}$ will follow a VARMA(p)(P)S(c)(Q)S process where $s = mS$ and $c = \left[p + 1 + \frac{q-p-1}{m}\right]$:

$$\boldsymbol{\varphi}_p(\mathrm{B})\boldsymbol{\Gamma}_P(\mathrm{B}^S)\boldsymbol{X}_T = \boldsymbol{\eta}_c(\mathrm{B})\boldsymbol{\Upsilon}_Q(\mathrm{B}^S)\boldsymbol{E}_T, \tag{5.33}$$

where $\boldsymbol{E}_T$ are a sequence of random variables with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{\Omega}_{\boldsymbol{E}}$.

Proof:

The matrix polynomial $\phi_p(\mathrm{B})$ can be written as

$$\phi_p(\mathrm{B}) = \boldsymbol{I} - \phi_1\mathrm{B} - \cdots - \phi_p\mathrm{B}^p = \prod_{i=1}^{p}(\boldsymbol{I} - \delta_i\mathrm{B}) \tag{5.34}$$

for δ_j's satisfying

$$\phi_j = (-1)^{j-1}\sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\cdots\sum_{i_{j-1}=i_{j-2}+1}^{p-1}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{j-1}}\boldsymbol{B}^{j-1},\ \phi_i = (-1)^{i-1}\sum_{j=1}^{p}\prod_{\ell=1}^{p-1}\delta_{j_\ell}, i = 1,\cdots,p-1 \text{ and}$$

$\phi_p = (-1)^p\prod_{j=1}^{p}\delta_j$ because

$$\prod_{i=1}^{p}(\boldsymbol{I} - \delta_i\mathrm{B}) = \boldsymbol{I} - \sum_{i=1}^{p}\delta_i\mathrm{B} + \sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\delta_{i_1}\delta_{i_2}\mathrm{B}^2 + \cdots +$$
$$+(-1)^{p-1}\sum_{i_1=1}^{p-1}\sum_{i_2=i_1+1}^{p-1}\cdots\sum_{i_{p-1}=i_{p-2}+1}^{p-1}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{p-1}}\mathrm{B}^{p-1} + (-1)^p\prod_{i=1}^{p}\delta_i\mathrm{B}^p.$$

Similarly, the matrix polynomial $\boldsymbol{\Phi}_P(\mathrm{B}^s)$ can be written as

$$\boldsymbol{\Phi}_P(\mathrm{B}^s) = \boldsymbol{I} - \boldsymbol{\Phi}_1\mathrm{B}^s - \cdots - \boldsymbol{\Phi}_P\mathrm{B}^{sP} = \prod_{i=1}^{P}(\boldsymbol{I} - \rho_i\mathrm{B}^s) \tag{5.35}$$

where ρ_j is satisfying

$$\boldsymbol{\Phi}_j = (-1)^{j-1}\sum_{i_1=1}^{P-1}\sum_{i_2=i_1+1}^{P-1}\cdots\sum_{i_{j-1}=i_{j-2}+1}^{P-1}\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{j-1}}B^{s(j-1)}, j = 1,\cdots,P-1 \text{ and } \boldsymbol{\Phi}_P = (-1)^P\prod_{j=1}^{P}\rho_j$$

because

$$\prod_{i=1}^{P}\left(I-\rho_i B^s\right)=I-\sum_{i=1}^{P}\rho_i B^s+\sum_{i_1=1}^{P-1}\sum_{i_2=i_1+1}^{P-1}\rho_{i_1}\rho_{i_2}B^{2s}+\cdots+$$
$$+(-1)^{P-1}\sum_{i_1=1}^{P-1}\sum_{i_2=i_1+1}^{P-1}\cdots\sum_{i_{P-1}=i_{P-2}+1}^{P-1}\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{P-1}}B^{s(P-1)}+(-1)^{P}\prod_{i=1}^{P}\rho_i B^{sP}.$$

The Equation (5.32) can be written as

$$\prod_{i=1}^{p}(I-\delta_i B)\prod_{i=1}^{P}\left(I-\rho_i B^s\right)x_t=\theta_q(B)\Theta_Q\left(B^s\right)a_t. \tag{5.36}$$

When we multiply both sides of Equation (5.36) by

$$(I-\delta_1 B)^{-1},$$

we get

$$(I-\delta_1 B)^{-1}\prod_{i=1}^{p}(I-\delta_i B)\prod_{i=1}^{P}\left(I-\rho_i B^s\right)x_t=(I-\delta_1 B)^{-1}\theta_q(B)\Theta_Q\left(B^s\right)a_t,$$

that is,

$$\prod_{i=2}^{p}(I-\delta_i B)\prod_{i=1}^{P}\left(I-\rho_i B^s\right)x_t=(I-\delta_1 B)^{-1}\theta_q(B)\Theta_Q\left(B^s\right)a_t. \tag{5.37}$$

When we multiply both sides of Equation (5.37) by

$$(I-\delta_2 B)^{-1},$$

we get

$$\prod_{i=3}^{p}(I-\delta_i B)\prod_{i=1}^{P}\left(I-\rho_i B^s\right)x_t=(I-\delta_2 B)^{-1}(I-\delta_1 B)^{-1}\theta_q(B)\Theta_Q\left(B^s\right)a_t.$$

We continue multiplying both sides of the new equation by $(I-\delta_i B)^{-1}$, $i=3,\cdots,p$ and then start multiplying both sides by $(I-\rho_i B)^{-1}$, $i=1,\cdots,P$. Finally, we obtain

$$x_t = \prod_{i=0}^{P-1}\left(I - \rho_{P-i}\mathrm{B}^s\right)\prod_{j=0}^{p-1}\left(I - \delta_{p-j}\mathrm{B}\right)^{-1}\theta_q(\mathrm{B})\Theta_Q\left(\mathrm{B}^s\right)a_t. \tag{5.38}$$

Then, we multiply both sides of Equation (5.38) by

$$\frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{i=1}^{P}\left(I - \rho_i^m\mathrm{B}^{\mathrm{mS}}\right)$$

and we have

$$\frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{i=1}^{P}\left(I - \rho_i^m\mathrm{B}^{\mathrm{mS}}\right)x_t$$
$$= \frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{i=1}^{P}\left(I - \rho_i^m\mathrm{B}^{mS}\right)\prod_{i=0}^{P-1}\left(I - \rho_{P-i}\mathrm{B}^s\right)^{-1}\prod_{j=0}^{p-1}\left(I - \delta_{p-j}\mathrm{B}\right)^{-1}\theta_q(\mathrm{B})\Theta_Q\left(\mathrm{B}^{\mathrm{ms}}\right)a_t.$$

Since $s = mS$,

$$\frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{i=1}^{P}\left(I - \rho_i^m\mathrm{B}^{\mathrm{mS}}\right)x_t$$
$$= \frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{j=0}^{p-1}\left(I - \delta_{p-j}\mathrm{B}\right)^{-1}\theta_q(\mathrm{B})\Theta_Q\left(\mathrm{B}^{\mathrm{mS}}\right)a_t.$$

By changing t to mT,

$$\frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{i=1}^{P}\left(I - \rho_i^m\mathrm{B}^{\mathrm{mS}}\right)x_{mT}$$
$$= \frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}\prod_{i=1}^{p}\left(I - \delta_i^m\mathrm{B}^m\right)\prod_{j=0}^{p-1}\left(I - \delta_{p-j}\mathrm{B}\right)^{-1}\theta_q(\mathrm{B})\Theta_Q\left(\mathrm{B}^{\mathrm{mS}}\right)a_{mT}. \tag{5.39}$$

Since $X_T = \frac{\left(1-\mathrm{B}^m\right)}{(1-\mathrm{B})}x_{mT}$, Equation (5.39) becomes

$$\prod_{i=1}^{p}\left(I-\delta_i^m \mathrm{B}\right)\prod_{i=1}^{P}\left(I-\rho_i^m \mathrm{B}^{S}\right)X_T$$
$$=\frac{\left(1-\mathrm{B}^m\right)}{\left(1-\mathrm{B}\right)}\prod_{i=1}^{p}\left(I-\delta_i^m \mathrm{B}^m\right)\prod_{j=0}^{p-1}\left(I-\delta_{p-j}\mathrm{B}\right)^{-1}\theta_q(\mathrm{B})\Theta_Q\left(\mathrm{B}^{mS}\right)a_{mT}$$

or

$$\varphi_p(\mathrm{B})\Gamma_P\left(\mathrm{B}^S\right)X_T=\frac{\left(1-\mathrm{B}^m\right)}{\left(1-\mathrm{B}\right)}\prod_{i=1}^{p}\left(I-\delta_i^m \mathrm{B}^m\right)\prod_{j=0}^{p-1}\left(I-\delta_{p-j}\mathrm{B}\right)^{-1}\eta_q(\mathrm{B})\Upsilon_Q\left(\mathrm{B}^S\right)a_{mT} \quad (5.40)$$

where $S=m/s$. When we look at the right-hand side of Equation (5.40), we can see the highest order of the non-seasonal MA parameters is $c=\left[q+(m-1)+p(m-1)\right]/m=p+1+\frac{q-p-1}{m}$. Therefore, equation (5.40) is given by

$$\varphi_p(\mathrm{B})\Gamma_P\left(\mathrm{B}^S\right)X_T=\eta_c(\mathrm{B})\Upsilon_Q\left(\mathrm{B}^S\right)E_T \quad (5.41)$$

where $S=m/s$ and $c=\left[p+1+\frac{q-p-1}{m}\right]$. Q.E.D.

Note that these orders are the maximum values and we assume that there is no hidden periodicity. Thus, we obtain a similar result to the univariate case that was proved by Wei (1978b).

5.7 Empirical Examples

5.7.1 Monthly Single-Family Housing Starts and Housing Sales for the Period of January 1965 through May 1975

For a real life example, let us consider two monthly U.S. housing series from January 1965 to May 1975: single-family housing starts as $x_{1,t}$ and housing sales as $x_{2,t}$. The source of the data is from the U.S. Census Bureau. The data are shown in Figure 5.1

with 125 observations. Both series have a seasonal behavior and there is a strong relationship between $x_{1,t}$ and $x_{2,t}$. This data set was analyzed by Hillmer and Tiao (1979) and they first fit a model for individual series. As a univariate model, they found that the multiplicative form best fits the data. By individual analysis, they found that a first order difference and a seasonal difference were required to obtain a stationary series.

Based on the sample autocorrelation and partial lag autocorrelation matrices, Hillmer and Tiao (1979) followed the traditional representation of a multiplicative seasonal ARMA model given in (5.2), and proposed the following multiplicative seasonal vector ARMA model:

$$(\boldsymbol{I}-\phi\mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^{12})\underset{2\times1}{\boldsymbol{x}_t}=(\boldsymbol{I}-\theta\mathrm{B})(\boldsymbol{I}-\Theta\mathrm{B}^{12})\underset{2\times1}{\boldsymbol{a}_t}. \quad (5.42)$$

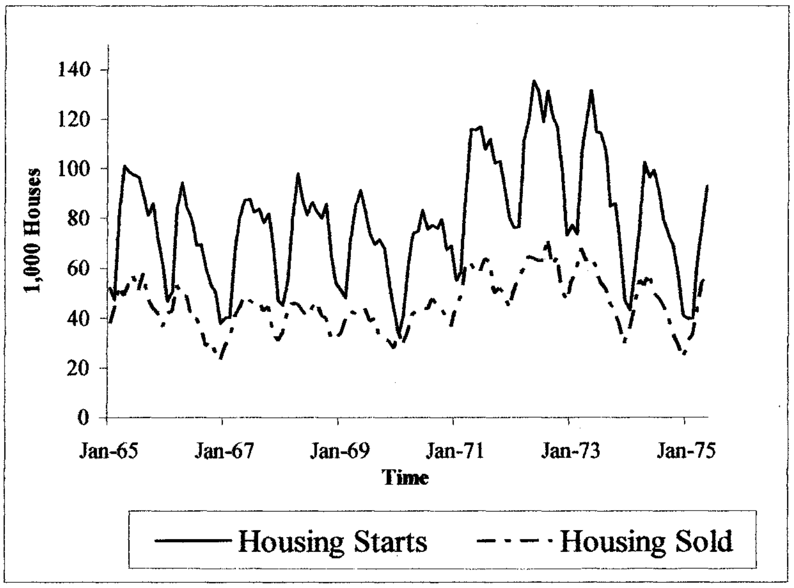


Figure 5.1 Monthly U.S. Single-family Housing Starts and Housing Sales from January 1965 to May 1975

To estimate the model, the time series software SCA and its exact likelihood estimation are used. Table 5.4 gives the estimates of the parameters ($\boldsymbol{\phi}$, $\boldsymbol{\theta}$, $\boldsymbol{\Theta}$) along with the covariance matrix, $\boldsymbol{\Omega}$ for $\boldsymbol{a}_t$. These values are approximately the same as the parameter estimates of Hillmer and Tiao.

Table 5.4 Parameter Estimates and Their Standard Deviations for Model (5.42)

$$\hat{\boldsymbol{\phi}} = \begin{pmatrix} 0.796 & 1.806 \\ (0.056) & (0.214) \\ -0.130 & -0.388 \\ (0.142) & (0.459) \end{pmatrix} \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} 1.316 & 0.938 \\ (0.014) & (0.192) \\ -0.357 & -0.060 \\ (0.122) & (0.495) \end{pmatrix} \quad \hat{\boldsymbol{\Theta}} = \begin{pmatrix} 0.788 & 0.005 \\ (0.071) & (0.113) \\ 0.082 & 0.696 \\ (0.051) & (0.081) \end{pmatrix} \quad \hat{\boldsymbol{\Omega}} = \begin{pmatrix} 36.780 & 7.258 \\ 7.258 & 15.652 \end{pmatrix}$$

Note that by switching the order of non-seasonal and seasonal polynomials, the model can also be represented as follows:

$$(\boldsymbol{I} - \boldsymbol{\phi}\mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^{12})\boldsymbol{x}_t = (\boldsymbol{I} - \boldsymbol{\Theta}\mathrm{B}^{12})(\boldsymbol{I} - \boldsymbol{\theta}\mathrm{B})\boldsymbol{a}_t. \qquad (5.43)$$

Representation (5.40) is different from the representation (5.42) of Hillmer and Tiao (1979) because the model is represented as $VARMA(0)_{12}(1)(1)_{12}(1)$ instead of the traditional $VARMA(1)(0)_{12}(1)(1)_{12}$ representation. The exact likelihood estimation of the representation (5.43) is given in Table 5.5.

Table 5.5 Parameter Estimates and Their Standard Deviations for Model (5.43)

$$\hat{\boldsymbol{\phi}} = \begin{pmatrix} 0.523 & 1.207 \\ (0.086) & (0.183) \\ 0.155 & 0.288 \\ (0.05) & (0.238) \end{pmatrix} \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} 1.043 & 0.395 \\ (0.062) & (0.226) \\ -0.058 & 0.588 \\ (0.088) & (0.247) \end{pmatrix} \quad \hat{\boldsymbol{\Theta}} = \begin{pmatrix} 0.763 & 0.091 \\ (0.064) & (0.099) \\ 0.035 & 0.745 \\ (0.035) & (0.066) \end{pmatrix} \quad \hat{\boldsymbol{\Omega}} = \begin{pmatrix} 35.962 & 7.660 \\ 7.660 & 15.837 \end{pmatrix}$$

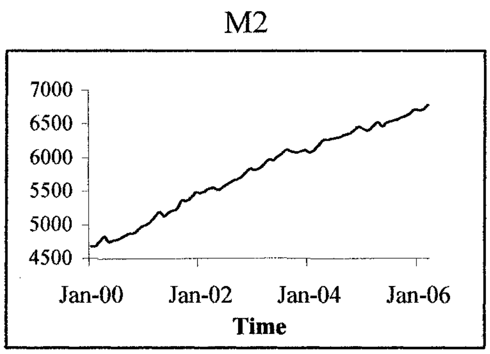
To evaluate the two different representations for (5.42) and (5.43), they are used to forecast the values for the next five periods and then compared with the actual values from June 1975 to October 1975 from the U.S. Bureau of Census. The comparison in terms of the square root average of the TMSFE is given in Table 5.6.

Table 5.6 The $M(\ell)$ Values of Two Different Representations for Monthly US Single-family Housing Starts and Housing Sales

	Model as $VARMA(1)(0)_{12}(1)(1)_{12}$	Model as $VARMA(0)_{12}(1)(1)_{12}(1)$
June 1975	7.276	6.990
July 1975	1.732	0.218
August 1975	3.392	1.646
September 1975	3.683	0.421
October 1975	11.481	8.606

5.7.2 Monthly M2 Stocks and Consumer Price Index for the Period of January 1959 through May 2006

Our second example uses two monthly U.S. money and price series from January 1959 to May 2006: M2 money stock as $x_{1,t}$ and the U.S. consumer price index for all urban consumers as $x_{2,t}$. The source of the first series is from the Board of Governors of the Federal Reserve System, and the second one is from the U.S. Department of Labor: Bureau of Labor Statistics. The sample size of the data is 569. In order to see the seasonal behavior of the series, the data are shown in Figure 5.2 with 69 observations. For forecasting purposes, we use 564 of the observations and keep the last 5 observations in order to compare them with their forecasts.



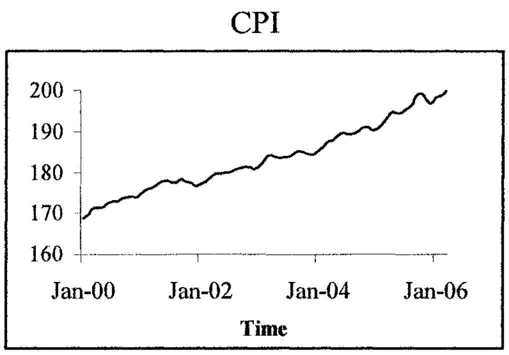


Figure 5.2 Monthly U.S. M2 Stocks and Consumer Price Index for the Period of January 2000 through May 2006

Both series have a seasonal behavior and an increasing trend over time. We need differencing to reduce them to stationary series.

To estimate the model, the time series software SCA and its likelihood estimation are used. After differencing of orders 1 and 12, Tables 5.7 and 5.8 give the schematic representations of cross correlation matrices and partial autoregression, respectively. These tables show that a possible model to represent this data set may be VARMA$(1)(0)_{12}(2)(1)_{12}$ or VARMA$(0)_{12}(1)(1)_{12}(2)$. That is, we propose the following multiplicative seasonal vector ARMA models:

VARMA$(1)(0)_{12}(2)(1)_{12}$:

$$(\boldsymbol{I}-\phi\mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^{12})\underset{2\times1}{\boldsymbol{x}_t}=(\boldsymbol{I}-\theta_1\mathrm{B}-\theta_2\boldsymbol{B}^2)(\boldsymbol{I}-\Theta\mathrm{B}^{12})\underset{2\times1}{\boldsymbol{a}_t}. \tag{5.44}$$

and VARMA$(0)_{12}(1)(1)_{12}(2)$:

$$(\boldsymbol{I}-\phi\mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^{12})\underset{2\times1}{\boldsymbol{x}_t}=(\boldsymbol{I}-\Theta\mathrm{B}^{12})(\boldsymbol{I}-\theta_1\mathrm{B}-\theta_2\boldsymbol{B}^2)\underset{2\times1}{\boldsymbol{a}_t}. \tag{5.45}$$

Table 5.7 Sample Cross Correlation Matrices of Monthly U.S. M2 Stocks and Consumer Price Index

```
LAGS   1 THROUGH   6

      + -      + -      + .      - .      . .      . .
      . +      . .      . .      . .      . .      - .

LAGS   7 THROUGH  12

      - +      . +      . .      - .      - .      - .
      . +      . .      . .      . +      . +      . -

LAGS  13 THROUGH  18

      - .      . +      . +      . .      + .      + .
      + -      . .      . +      - .      . .      . .

LAGS  19 THROUGH  24

      + .      + .      . .      . .      + .      - .
      . .      . .      . .      . .      . -      + -
SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
+ DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
- DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
. DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION
```

Table 5.8 Sample Partial Cross Correlations of Monthly U.S. M2 Stocks and Consumer Price Index

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
x1	+-	.-	..	-.	..	..	-.	..	..	-.	-.	-.
x2	.+	.-	..	-.	..	..	..	-.	++	.+	..	.-

+ is > 2*std error, - is < -2*std error, . is between

Estimates of the parameters (ϕ, θ_1, θ_2, Θ) along with the covariance matrix, Ω for a_t are given in Table 5.9 for Model in (5.44).

Table 5.9 Parameter Estimates and Their Standard Deviations for Model (5.44)

$\hat{\boldsymbol{\phi}} = \begin{pmatrix} 0.321 & 85.683 \\ (0.168) & (12.605) \\ 0.012 & -0.893 \\ (0.003) & (0.159) \end{pmatrix}$	$\hat{\theta}_1 = \begin{pmatrix} -0.031 & -84.015 \\ (0.176) & (12.172) \\ 0.014 & -1.334 \\ (0.003) & (0.159) \end{pmatrix}$	$\hat{\theta}_2 = \begin{pmatrix} 0.179 & -36.523 \\ (0.107) & (6.332) \\ 0.007 & -0.420 \\ (0.001) & (0.082) \end{pmatrix}$
$\hat{\boldsymbol{\Theta}} = \begin{pmatrix} 0.628 & -3.271 \\ (0.036) & (1.608) \\ -0.0045 & 0.772 \\ (0.007) & (0.031) \end{pmatrix}$	$\hat{\boldsymbol{\Omega}} = \begin{pmatrix} 118.163 & 0.156 \\ 0.156 & 0.068 \end{pmatrix}$	

Similarly, by switching the order of the non-seasonal and seasonal polynomials, the likelihood estimation of Model in (5.45) is given in Table 5.10.

Table 5.10 Parameter Estimates and Their Standard Deviations for Model (5.45)

$\hat{\boldsymbol{\phi}} = \begin{pmatrix} 0.290 & -70.765 \\ (0.158) & (10.768) \\ 0.014 & -0.830 \\ (0.003) & (0.153) \end{pmatrix}$	$\hat{\theta}_1 = \begin{pmatrix} -0.067 & -70.886 \\ (0.167) & (10.516) \\ 0.017 & -1.265 \\ (0.003) & (0.154) \end{pmatrix}$	$\hat{\theta}_2 = \begin{pmatrix} 0.175 & -30.603 \\ (0.099) & (5.401) \\ 0.008 & -0.385 \\ (0.002) & (0.079) \end{pmatrix}$
$\hat{\boldsymbol{\Theta}} = \begin{pmatrix} 0.692 & 1.395 \\ (0.026) & (1.346) \\ 0.002 & 0.729 \\ (0.0003) & (0.027) \end{pmatrix}$	$\hat{\boldsymbol{\Omega}} = \begin{pmatrix} 118.778 & 0.080 \\ 0.080 & 0.068 \end{pmatrix}$	

To evaluate the two different representations for (5.44) and (5.45), they are used to forecast the values for the next twelve periods and then compared with the actual values from January 2006 to May 2006. The comparison in terms of the square root average of the TMSFE is given in Table 5.11.

Table 5.11 The $M(\ell)$ Values of Two Different Representations for Monthly U.S. M2 Stocks and Consumer Price Index

	Model as $VARMA(1)(0)_{12}(2)(1)_{12}$	Model as $VARMA(0)_{12}(1)(1)_{12}(2)$
January 2006	12.4	12.5
February 2006	17.6	18.1
March 2006	14.0	16.1
April 2006	10.7	18.9
May 2006	24.9	17.6

The difference in the forecasting performance between the two representations of the multiplicative vector model is very significant. The representation given in (5.44) gives much better forecasts. The AIC for representation (5.44) is 2.138, which is slightly smaller than the AIC for representation (5.45) at 2.145. Thus for this data set, we propose the traditional representation of the multiplicative model given in (5.44).

5.7.2.1 Analysis of Temporal Aggregates

The two quarterly U.S. money and price series from the first quarter of 1959 to last quarter of 2005 are the M2 money stock as $X_{1,T}$ and U.S. consumer price index for all urban consumers as $X_{2,T}$; these are obtained from the monthly series. The sample size reduces one third which makes N = 188. The first quarter of 2006 will be used for forecasting purposes. The time series plot of the data is given in Figure 5.3. The graph shows that both series have a seasonal behavior and an increasing trend over time.

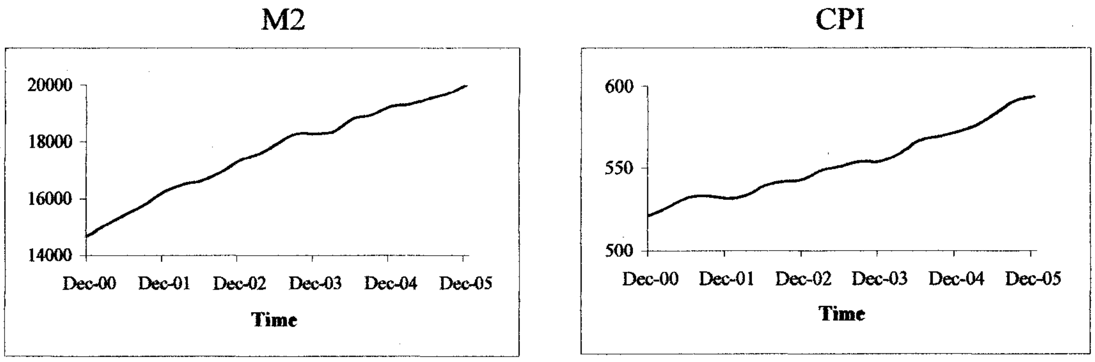


Figure 5.3 Quarterly U.S. M2 Stocks and the Consumer Price Index for the Period Q4_2000 through Q4_2005

To estimate the model, the time series software SCA and its likelihood estimation are used. After the differencing of orders 1 and 4, Tables 5.11 and 5.12 present the schematic representations of cross correlation matrices and partial autoregression, respectively. These tables show that possible models to represent this data set may be VARMA(1)(0)$_4$(1)(1)$_4$ or VARMA(0)$_4$(1)(1)$_4$(1). That is, we propose the following multiplicative seasonal vector ARMA models:

VARMA(1)(0)$_4$(1)(1)$_4$:

$$(\boldsymbol{I}-\varphi \mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^4)\underset{2\times 1}{\boldsymbol{X}_T}=(\boldsymbol{I}-\eta \mathrm{B})(\boldsymbol{I}-\Gamma \mathrm{B}^4)\underset{2\times 1}{\boldsymbol{E}_T}. \tag{5.46}$$

and

VARMA(0)$_4$(1)(1)$_4$(1):

$$(\boldsymbol{I}-\varphi \mathrm{B})(1-\mathrm{B})(1-\mathrm{B}^4)\underset{2\times 1}{\boldsymbol{X}_T}=(\boldsymbol{I}-\Gamma \mathrm{B}^4)(\boldsymbol{I}-\eta \mathrm{B})\underset{2\times 1}{\boldsymbol{E}_T}. \tag{5.47}$$

Table 5.12 Sample Cross Correlation Matrices of Quarterly U.S. M2 Stocks and the Consumer Price Index

```
LAGS  1 THROUGH  6

     + -     . .     . .     - .     . .     + .
     - +     - .     . +     . -     . .     . .

LAGS  7 THROUGH 12

     . .     . .     . .     . .     . .     . .
     . .     . .     . -     . .     . -     . .

LAGS 13 THROUGH 18

     + .     . .     . -     . -     . .     . .
     . .     . .     . .     + .     + .     . .

LAGS 19 THROUGH 24

     . .     . .     . .     . .     . .     . .
     . .     . .     . .     . .     . .     . .

LAGS 25 THROUGH 30

     . .     . .     . .     . .     . .     . .
     . .     . .     . .     . .     . .     . .
```

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
+ DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
- DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
. DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION

Table 5.13 Sample Partial Cross Correlations of Quarterly U.S. M2 Stocks and the Consumer Price Index

Variable/ Lag	1	2	3	4	5	6	7	8	9	10	11	12
X1	+.	-+	.-	..	..	..	..	..	..	..	..	..
X2	.+	..	.-	..	..	..	..	..	..	..	..	..

Estimates of the parameters (φ, η, Γ) along with the covariance matrix, Ω_E for E_T are given in Tables 5.14 and 5.15 for the models in (5.46) and (5.47), respectively.

Table 5.14 Parameter Estimates and Their Standard Deviations for Model (5.46)

$\hat{\varphi}=\begin{pmatrix} 0.738 & 19.142 \\ (0.098) & (4.165) \\ 0.008 & 0.404 \\ (0.003) & (0.107) \end{pmatrix}$	$\hat{\eta}=\begin{pmatrix} 0.345 & 30.186 \\ (0.118) & (5.787) \\ 0.013 & -0.227 \\ (0.003) & (0.114) \end{pmatrix}$
$\hat{\Gamma}=\begin{pmatrix} 0.931 & 3.859 \\ (0.047) & (1.885) \\ -0.006 & 0.857 \\ (0.001) & (0.044) \end{pmatrix}$	$\hat{\Omega}_E=\begin{pmatrix} 3485.77 & -8.72 \\ -8.72 & 1.78 \end{pmatrix}$

Table 5.15 Parameter Estimates and Their Standard Deviations for Model (5.47)

$\hat{\varphi}=\begin{pmatrix} 0.624 & 19.197 \\ (0.104) & (4.314) \\ 0.009 & 0.479 \\ (0.003) & (0.105) \end{pmatrix}$	$\hat{\eta}=\begin{pmatrix} 0.212 & 30.178 \\ (0.121) & (5.557) \\ 0.014 & -0.108 \\ (0.003) & (0.117) \end{pmatrix}$
$\hat{\Gamma}=\begin{pmatrix} 0.870 & 1.261 \\ (0.040) & (1.724) \\ 0.0099 & 0.886 \\ (0.008) & (0.037) \end{pmatrix}$	$\hat{\Omega}_E=\begin{pmatrix} 3569.96 & -9.33 \\ -9.33 & 1.76 \end{pmatrix}$

To evaluate the two different representations for (5.46) and (5.47), we obtain the first step-ahead forecasts and then compared them with the actual values for Q1_2006. The comparison, in terms of the square root average of the TMSFE, is given in Table 5.16.

Table 5.16 The $M(\ell)$ Values of Two Different Representations for Quarterly U.S. M2 Stocks and the Consumer Price Index

	Model as $VARMA(1)(0)_4(1)(1)_4$	Model as $VARMA(0)_4(1)(1)_4(1)$
Q1_2006	26.50	35.53

The difference in forecasting performance between the two representations of the multiplicative vector model is very significant. The representation given in (5.46) gives much better forecasts. The AIC for representation (5.46) is 8.848, which is slightly smaller than the AIC for representation (5.47) at 8.859. Thus for this data set, we propose the traditional representation of multiplicative model (5.46) with the parameters given in Table 5.14. This result is consistent with Proposition 5.1. The order of seasonal and non-seasonal matrix polynomials in the best representations for the basic series and the aggregate series are the same.

CHAPTER 6

CONCLUDING REMARKS

Time series often contains observations of several variables, and multivariate vector time series processes are used to study the relationship between these variables. The time series data used are typically sums or averages of data that is frequently generated more than the reporting interval. The study of temporal aggregation of multivariate processes is important because many properties of interest, such as causality and cointegration, can only be studied through multivariate processes.

After an introduction of basic concepts for vector processes, we analyzed some properties of vector time series under temporal aggregation and derived some vector ARIMA models for temporal aggregates for some commonly used time series processes. The results show that temporal aggregation affects the model structure of vector ARIMA models. Temporal aggregation often results in a moving average term. Even though the data generation process for the basic series is a simple vector AR, we have a vector ARMA model after temporal aggregation. This affects many multivariate time series analyses. Based on this fact, we developed proper test statistics for testing cointegration, causality, and others when aggregate time series are used.

We have demonstrated that although cointegration remained upon aggregation, the error correction representations for the basic series and aggregated series are no longer the same. To see the effect of aggregation on an existing likelihood based test, we performed a simulation study on a simple cointegrated vector AR(1) model. As the aggregation period m increases, the test indicates no cointegration, and this contradicts the theoretical result that we have proved that cointegration remained upon aggregation. Therefore, the test statistic to test cointegration in the system needs to be modified for aggregate data. Based on the new error correction representation, we develop a new test statistic and obtain its limiting distribution for time series aggregates.

In the literature, there are many studies on the effect of temporal aggregation on the Granger causality. However, these studies do not take into consideration the fact that the form of the vector time series model is changed after aggregation. We have shown that the non-causality conditions are not the same for the basic and aggregate series. Through the vector time series model of aggregates, we have shown the distortion effect of aggregation on causality, which is consistent with the results proved by Tiao and Wei (1976), and Wei (1982) in terms of a distributed lag model.

In a vector autoregressive process, the Granger non-causality of one set of variables for another set of variables is characterized by no constraints on the autoregressive coefficients. If the process is stationary, the test for non-causality is usually performed using Wald (or likelihood ratio) tests which are asymptotically chi-squared. Although Wald tests for Granger causality will not maintain their usual asymptotic properties in general for cointegrated systems, they maintain their asymptotic chi-square distribution for bivariate systems (Lütkepohl and Reimers, 1992). These results emphasize that before performing the Granger causality test, one should investigate the existence of the unit roots or cointegration in the system. Under temporal aggregation, the Wald test or likelihood ratio test for testing non-causality in cointegrated systems is the same. We only consider the structure change in the vector autoregressive models. Because of this change, the non-causality conditions are much more complicated than the conditions for the basic series. Mosconi and Giannini (1992) suggest a likelihood ratio test which is more efficient by imposing the cointegration constraints under both the null and the alternative hypotheses. Therefore, we used their approach to test causality in cointegrated systems for aggregates. Since the error correction model for aggregates is different from the error correction model of the basic series aggregates given in Mosconi and Giannini (1992), the testing procedure needed to be corrected for the aggregates. We developed the new testing procedure to test non-causality in cointegrated systems for aggregates. We derived the test statistic that is used to test non-causality in cointegrated systems when aggregates are used. It is shown that the limiting distribution of the test statistics is the same as that of basic series. We performed a simulation study to see how well our testing procedure works. We used a cointegrated bivariate vector AR(1) model where the second variable does not cause the first variable. Our simulation study shows the distortion effect of temporal aggregation on the causal relationship in cointegrated system. Temporal aggregation changes the non-causal relationship to causal one. When the sample size increases, the probability of rejecting the non-causality in cointegrated system also increases. With an increasing aggregation period, there is a loss in the power of the test because of the decrease in sample size. The power of the test approaches 1.000 when the sample size increases. Therefore, we recommend using the test which is designed specifically for aggregates in order to test non-causality in cointegrated systems.

In a time series analysis, we often see periodic behavior and use multiplicative models to describe this phenomenon. In a univariate time series analysis, the order of non-seasonal and seasonal parameters does not matter because of the commutative property of scalar multiplication; however, the order becomes important for the multivariate case. In general, the matrix multiplication is non-commutative. In order to see the effect of the order change in non-seasonal and seasonal parameters, we looked at the likelihood functions of two representations. In the first representation, the traditional way is used, that is, the non-seasonal matrix polynomial is placed in front of the seasonal matrix polynomial. In the second representation, we place the seasonal matrix polynomial in front of non-seasonal matrix polynomial. We have shown that the maximum likelihood estimation results are different for the two representations.

Through a detailed simulation study, we examine the impacts of different representations on parameter estimation, forecasting, and causality. It has been shown that the order of non-seasonal and seasonal parameter matrix polynominals in the representation is important. They are not interchangeable. We consider three types of information criteria: the Akaike Information Criterion, the Hannan and Quinn Information Criterion, and the Schwarz Information Criteria in selecting the best representation of a seasonal vector time model. All three information criteria reach the same result. The smaller the information criterion is, the better the representation of the multiplicative model. Finally, we have extended the univariate seasonal time series result of Wei (1978b) to the multiplicative seasonal vector model.

One of the important issues in time series analysis is the decision of the time unit that will be used. The data sources present many choices. For instance, one can choose to work with monthly, quarterly or annual data. As the number of aggregation period becomes larger, the model structure tends to simplify. However, many studies show that aggregation causes loss of information. Therefore, an interesting problem to study is to decide an optimal time unit that should be used in time series analysis.

In this research we consider the cointegration test for aggregates with no drift and trend terms in the series. The work can be extended to investigate the effects of aggregation on the likelihood based cointegration test when there is a drift and/or a trend in the series. Moreover, the cointegration test for aggregate series with restrictions on the parameters can also be considered.

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APPENDIX

FORTRAN PROGRAMS

We now present the FORTRAN programs used to compute some of the test statistics considered in our study.

1. Cointegration Test, Estimation of ECM Parameters and Test Non-Causality in Cointegrated Systems for Aggregation in VAR(2): The user has to provide dimension of the system k, VAR and VMA orders, P and Q and sample size N, differenced aggregated series W, the aggregated time series X, in an input file. The logical MEAN value should set to F. First two outputs give the trace statistic by using ECM with and without considering aggregation effect. Next output gives the MLE estimators of the ECM parameters for aggregates. At last the non-causality test result for the null hypothesis that is first variable does not cause the second one. To be able to test second variable does not cause the first one, user need to change U matrix as UO and UO matrix as U. The code is the following:

```
************************************************************
*           FORTRAN PROGRAM TO TEST COINTEGRATION          *
*      FIND MLES AND TEST NON-CAUSALITY FOR AGGREGATES     *
************************************************************
      USE Numerical_Libraries

*     .. Parameters ..

      INTEGER          KMAX, IK, IPMAX, IQMAX, NMAX, NPARMX, ICM, LWORK,

     +                  LIW

      PARAMETER         (KMAX=3,IK=KMAX,IPMAX=3,IQMAX=3,NMAX=500,

     +          NPARMX=(IPMAX+IQMAX)*KMAX*KMAX+KMAX,ICM=NPARMX,

     +          LWORK=3000,LIW=500)
```

```
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5)
*     .. Local Scalars ..
      DOUBLE PRECISION CGETOL, RLOGL
      INTEGER          I,IFAIL, IP, IPRINT, IQ, ISHOW, J, K, MAXCAL, N,
     +                 NITER, NPAR
      LOGICAL          EXACT, MEAN
*     .. Local Arrays ..
      DOUBLE PRECISION CM(ICM,NPARMX), G(NPARMX), PAR(NPARMX),
     +                 QQ(IK,KMAX), V(IK,NMAX), W(IK,NMAX), WORK(LWORK)
      REAL  X(170,2),DX(168,2),DDX(2,2),TDX(2,2),TXX(2,2),
     +      GINV(2,2),XXINV(2,2),TXD(2,2),XD(2,168),TX(2,168),
     +      S1(2,2),S2(2,2),S3(2,2),S4(2,2),Y(168,2),XT(168,2),
     +      MX(168,2),XM(2,168),MMX(2,2),MINV(2,2),MDX(2,2),
     +      XMY(2,2),YMX(2,2),T00(2,2),T0P(2,2),TP0(2,2),TPP(2,2),
     +      T1(2,2),T2(2,2),T3(2,2),S5(2,2),DMX(2,2),
     +      A(168,2),TA(2,168),YDA(2,2),YAD(2,2),TAA(2,2),ITAA(2,2),
     +      A1(2,2),S00(2,2),Z(2,2),IZ(2,2),TAY(2,2),TYA(2,2),A2(2,2),
     +      S0P(2,2),Z1(2,2),Z2(2,2),Z3(2,2),Z4(2,2),XMA(2,2),SA1(2,2),
     +      SA2(2,2),S0E(2,2),H0E(2,2),SH0E(2,2),XAA(2,2),SPE(2,2),
     +      HPE(2,2),SHPE(2,2),SHEP(2,2),MXA(2,2),SA3(2,2),SEE(2,2),
     +      HEE(2,2),IHEE(2,2),HEP(2,2),SA4(2,2),F0P(2,2),FP0(2,2),
     +      SF0P(2,2),SFP0(2,2),SF00(2,2),HE0(2,2),ISF00(2,2),SA5(2,2),
     +      F00(2,2),FPP(2,2),SFPP(2,2),ISFPP(2,2),AS1(2,2),AS2(2,2),
     +      AS3(2,2),ETXX(2,2),EVTXX(2,2),TEV(2,2),TE(2,2),M11(2,2),
     +      SE(2,2),TEM(2,2),SQM(2,2),XET(2,2),BET(2,1),ET(1,2),
     +      PHI(2,2),AT(1,2),ATA,ALPHA(2,1),E(2,2),ABET(2,2),
     +      ABEH(2,2),THE(2,2),THETA(2,2),GAM(2,2),G1(2,2),G2(2,2),
```

```
+       G3(2,2),V1(2,168),V2(2,168),V3(2,168),VAR(2,168),
+       TVAR(168,2),VV(2,2),SIGMA(2,2),YY(2,1),F1(2,1),F2(2,1),
+       F3(2,1),FORE(2,1),XF(2,1),RES(2,1),F0(2,1),FERR(2,1),
+       U(2,1),UO(2,1),TUU(1,2),TUO(1,2),
&       SAA2(2,2),SAA3(2,2),SKB(2,1),SBK(1,2),S001(1,2),
&       SKK1(2,2),SAA1(1,2),SKKB(2,2),ISKKB(2,2),SAKB(1,2),
&       SKAB(2,1),SAK(1,2),SAK3(1,2),SBA(1,2),
&       SAK4(1,2),SAK5(2,1),SAK6(2,2),SAK7(2,2),SAK8(2,2)
 REAL U(2,1),UO(2,1),TUU(1,2),TUO(1,2),
&      Y1(1,2),Y3(2,1),Y4(1,2),Y5(2,2),Y6(2,2),
&      IY(2,2),W1(1,2), P1(1,2),P2(2,1),P3(2,2),P4(2,2),
&      P5(2,2),W3(1,2),
&      Y7(2,1),Y8(1,2),Y9(2,2),Y10(2,2),Y11(2,2),Y12(2,2),
&      Y13(2,2),Y14(2,2),Y15(2,2),
&      TBET(1,2),B1(1,1),B2,B3(2,2),B4(2,2),B5(2,2),
&      B6(1,2),ETX(2,2),EVTX(2,2),
&      ALPS(2,1), TALP(1,2),D1(1,2),D3(2,2),
&      DEV(2,2),DEX(2,2),DE(2,2),ED(2,2),DQM(2,2),XED(2,2),
&      DBET(1,2),BED(2,1),SED(2,2),DEM(2,2),ZZ3(2,2),ZZ4(2,2),
&      SAA1(2,2),SAA2(2,2),SAA3(2,2)
 COMPLEX    EVAL(2), EVEC(2,2), EVALA(2), EVECA(2,2),
&           EVALT(2), EVECT(2,2),EVECAR(2,2),EVALAR(2)
 INTEGER          IW(LIW)
 LOGICAL          PARHLD(NPARMX)
 OPEN (NIN, FILE='C:\WAGE_AR2.TXT',STATUS='OLD')
 OPEN (NOUT,FILE='C:\RESULT.TXT',STATUS='REPLACE')
*     .. Executable Statements ..
*     Skip heading in data file
```

```
      READ (NIN,*)
      READ (NIN,*) K, IP, IQ, N, MEAN
      CALL X04ABF(1,NOUT)
      WRITE (NOUT,*)
      IF (K.GT.0 .AND. K.LE.KMAX .AND. IP.GE.0 .AND. IP.LE.IPMAX .AND.
     +    IQ.GE.0 .AND. IQ.LE.IQMAX) THEN
         NPAR = (IP+IQ)*K*K
         IF (MEAN) NPAR = NPAR + K
         IF ((N.LE.NMAX) .AND. (NPAR.LE.NPARMX)) THEN
            DO 20 I = 1, NPAR
               PAR(I) = 0.0e0
               PARHLD(I) = .FALSE.
   20       CONTINUE
*
*           Set all elements of Q to zero to use covariance matrix
*           between the K time series as the initial estimate of the
*           covariance matrix
*
            DO 60 J = 1, K
               DO 40 I = J, K
                  QQ(I,J) = 0.0
   40          CONTINUE
   60       CONTINUE
            DO 80 I = 1, K
               READ (NIN,*) (W(I,J),J=1,N)
   80       CONTINUE
            EXACT = .TRUE.
*           ** Set IPRINT .GT. 0 to obtain intermediate output
```

```
            IPRINT = -1
            CGETOL = 0.0001e0
            MAXCAL = 40*NPAR*(NPAR+5)
            ISHOW = 2
            IFAIL = -1
*
            CALL G13DCF(K,N,IP,IQ,MEAN,PAR,NPAR,QQ,IK,W,PARHLD,EXACT,
     +                  IPRINT,CGETOL,MAXCAL,ISHOW,NITER,RLOGL,V,G,CM,
     +                  ICM,WORK,LWORK,IW,LIW,IFAIL)
*
         END IF
      END IF
C
      READ(NIN,*)
      READ(NIN,*) ((X(I,J),J=1,K),I=1,N+2)
C
C              FIND THE TRANSPOSE OF THE ERROR MATRIX
      DO 120 I=1,N
            DO 100 J=1,K
                  A(I,J)=V(J,I)
100         CONTINUE
120   CONTINUE
      A(1,1)=0
      A(1,2)=0
C              FIND THE DIFFERENCES (1-B)X
      DX(1,1)=0
      DX(1,2)=0
      DO 140 J=2,N
```

```
      DX(J,1)=X(J+2,1)-X(J+1,1)
      DX(J,2)=X(J+2,2)-X(J+1,2)
140   CONTINUE
      DO 160 IJ=1,N
        Y(IJ,1)=X(IJ,1)
        Y(IJ,2)=X(IJ,2)
160   CONTINUE
C                  FIND THE DIFFERENCES (1-B)**2X
      MX(1,1)=0
      MX(1,2)=0
      MX(2,1)=0
      MX(2,2)=0
      DO 180 J=3,N
        MX(J,1)=X(J+1,1)-X(J,1)
        MX(J,2)=X(J+1,2)-X(J,2)
180   CONTINUE
C           CALCULATION OF THE TRACE TEST STATISTIC FOR AGGREGATES
      CALL TRNRR (N, K, DX, N, K, N, XD, K)
      CALL MRRRR (K, N, XD, K, N, K, DX, N, K, K,DDX, K)
      CALL MRRRR (K, N, XD, K, N, K, Y, N, K, K,TDX, K)
      CALL TRNRR (N, K, Y, N, K, N, TX, K)
      CALL MRRRR (K, N, TX, K, N, K, Y, N, K, K,TXX, K)
      CALL TRNRR (N, K, MX, N, K, N, XM, K)
      CALL MRRRR (K, N, XM, K, N, K, MX, N, K, K,MMX, K)
      CALL LINRG (K, MMX, K, MINV, K)
      CALL MRRRR (K, N, XM, K, N, K, DX, N, K, K,MDX, K)
      CALL TRNRR (K, K, MDX, K, K, K, DMX, K)
      CALL MRRRR (K, N, XM, K, N, K, Y, N, K, K,XMY, K)
```

```
      CALL TRNRR (K, K, XMY, K, K, K, YMX, K)
      CALL MRRRR (K, K, DMX, K, K, K, MINV, K, K, K, S1, K)
      CALL MRRRR (K, K, S1, K, K, K, MDX, K, K, K, S2, K)
      TOO=DDX-S2
      CALL MRRRR (K, K, S1, K, K, K, XMY, K, K, K, S3, K)
      TOP=TDX-S3
      CALL TRNRR (K, K, TOP, K, K, K, TPO, K)
      CALL MRRRR (K, K, YMX, K, K, K, MINV, K, K, K, S4, K)
      CALL MRRRR (K, K, S4, K, K, K, XMY, K, K, K, S5, K)
      TPP=TXX-S5
      CALL LINRG (K, TOO, K, GINV, K)
      CALL LINRG (K, TPP, K, XXINV, K)
      CALL MRRRR (K, K, TPO, K, K, K, GINV, K, K, K, T1, K)
      CALL MRRRR (K, K, T1, K, K, K, TOP, K, K, K, T2, K)
      CALL MRRRR (K, K, T2, K, K, K, XXINV, K, K, K, T3, K)
      CALL EVCRG (K, T3, K, EVAL, EVEC, K)
      CALL WRCRN ('EVAL', 1, K, EVAL, 1, 0)
      CALL WRCRN ('EVEC', K, K, EVEC, K, 0)
C
      TRACE=-N*(LOG(1-EVAL(1))+LOG(1-EVAL(2)))
      TRACE1=-N*(LOG(1-EVAL(2)))
C
      WRITE(*,*) "TRACE(H=0):",TRACE
      WRITE(*,*) "TRACE(H=1):",TRACE1
C
      CALL TRNRR (N, K, A, N, K, N, TA, K)
      CALL MRRRR (K, N, XD, K, N, K, A, N, K, K,YDA, K)
      CALL TRNRR (K, K, YDA, K, K, K, YAD, K)
```

```
CALL MRRRR (K, K, DMX, K, K, K, MINV, K, K, K,SA1, K)
CALL MRRRR (K, N, XM, K, N, K, A, N, K, K,XMA, K)
CALL MRRRR (K, K, S1, K, K, K, XMA, K, K, K,SOE, K)
HOE=YDA-SOE
CALL MRRRR (K, N, Y, K, N, K, A, N, K, K,XAA, K)
CALL MRRRR (K, K, YMX, K, K, K, MINV, K, K, K,SA2, K)
CALL MRRRR (K, K, SA2, K, K, K, XMA, K, K, K,SPE, K)
HPE=XAA-SPE
CALL TRNRR (K, K, HPE, K, K, K, HEP, K)
CALL MRRRR (K, N, TA, K, N, K, A, N, K, K,TAA, K)
CALL TRNRR (K, K, XMA, K, K, K, MXA, K)
CALL MRRRR (K, K, MXA, K, K, K, MINV, K, K, K,SA3, K)
CALL MRRRR (K, K, SA3, K, K, K, XMA, K, K, K,SEE, K)
HEE=TAA-SEE
CALL LINRG (K, HEE, K, IHEE, K)
CALL MRRRR (K, K, HOE, K, K, K, IHEE, K, K, K,SA4, K)
CALL MRRRR (K, K, SA4, K, K, K, HEP, K, K, K,FOP, K)
CALL TRNRR (K, K, FOP, K, K, K, FPO, K)
SFOP=(TOP-FOP)/N
SFPO=(TPO-FPO)/N
CALL TRNRR (K, K, HOE, K, K, K, HEO, K)
CALL MRRRR (K, K, SA4, K, K, K, HEO, K, K, K,FOO, K)
SFOO=(TOO-FOO)/N
CALL LINRG (K, SFOO, K, ISFOO, K)
CALL MRRRR (K, K, HPE, K, K, K, IHEE, K, K, K,SA5, K)
CALL MRRRR (K, K, SA5, K, K, K, HEP, K, K, K,FPP, K)
SFPP=(TPP-FPP)/N
CALL LINRG (K, SFPP, K, ISFPP, K)
```

```
      CALL MRRRR (K, K, SFP0, K, K, K, ISF00, K, K, K,AS1, K)
      CALL MRRRR (K, K, AS1, K, K, K, SF0P, K, K, K,AS2, K)
      CALL MRRRR (K, K, ISFPP, K, K, K, AS2, K, K, K,AS3, K)
      CALL EVCRG (K, AS3, K, EVALA, EVECA, K)
      CALL WRCRN ('EVAL_AGGREGATE', 1, K, EVALA, 1, 0)
      CALL WRCRN ('EVEC_AGGREGATE', K, K, EVECA, K, 0)
C
      TRACE_AGG=-N*(LOG(1-EVALA(1))+LOG(1-EVALA(2)))
      TRACE1_AGG=-N*(LOG(1-EVALA(2)))
C
      WRITE(*,*) "TRACE_AGG(H=0):",TRACE_AGG
      WRITE(*,*) "TRACE_AGG(H=1):",TRACE1_AGG
C                 FIND THE ESTIMATES OF ECM
      CALL EVCRG (K, SFPP, K, EVALT, EVECT, K)
      ETXX(1,1)=EVALT(1)
      ETXX(1,2)=0.0
      ETXX(2,1)=0.0
      ETXX(2,2)=EVALT(2)
C
      EVTXX(1,1)=EVECT(1,1)
      EVTXX(1,2)=EVECT(1,2)
      EVTXX(2,1)=EVECT(2,1)
      EVTXX(2,2)=EVECT(2,2)
C
      CALL TRNRR (K, K, EVTXX, K, K, K, TEV, K)
      CALL MRRRR (K, K, EVTXX, K, K, K, ETXX, K, K, K, TE, K)
      CALL MRRRR (K, K, TE, K, K, K, TEV, K, K, K, M11, K)
C                 FIND THE (-0.5)TH POWER OF M11
```

```
      SE(1,1)=ETXX(1,1)**(-0.5)
      SE(1,2)=ETXX(1,2)
      SE(2,1)=SE(1,2)
      SE(2,2)=ETXX(2,2)**(-0.5)
      CALL MRRRR (K, K, EVTXX, K, K, K, SE, K, K, K, TEM, K)
      CALL MRRRR (K, K, TEM, K, K, K, TEV, K, K, K, SQM, K)
C
      E(1,1)=EVECA(1,1)
      E(1,2)=EVECA(1,2)
      E(2,1)=EVECA(2,1)
      E(2,2)=EVEC(2,2)
C
      CALL MRRRR (K, K, SQM, K, K, K, E, K, K, K , XET, K)
      BET(1,1)=XET(1,2)
      BET(2,1)=XET(2,2)
      CALL WRRRN ('BETA', K, 1, BET, K, 0)
C                       ESTIMATION OF THE ADJUSTMENT VECTOR
      CALL TRNRR (K, 1, BET, K, 1, K, ET, 1)
      CALL MRRRR (K, K, SFOP, K, K, 1, BET, K, K, 1, ALPHA, K)
      CALL WRRRN ('ALPHA', K, 1, ALPHA, K, 0)
C                        ESTIMATION OF PHI
      CALL MRRRR (K, 1, ALPHA, K, 1, K, ET, 1, K, K, PHI, K)
      CALL WRRRN ('PHI', K, K, PHI, K, 0)
C                        ESTIMATION OF THETA
      CALL MRRRR (K, 1, ALPHA, K, 1, K, ET, 1, K, K, ABET, K)
      CALL MRRRR (K, K, ABET, K, K, K, HPE, K, K, K, ABEH, K)
      THE=ABEH-HOE
      CALL MRRRR (K, K, THE, K, K, K, IHEE, K, K, K, THETA, K)
```

```
      CALL WRRRN ('THETA', K, K, THETA, K, 0)
C                         ESTIMATION OF GAMMA
      CALL MRRRR (K, K, PHI, K, K, K, XMY, K, K, K, G1, K)
      CALL MRRRR (K, K, THETA, K, K, K, MXA, K, K, K, G2, K)
      G3=DMX-G1+G2
      CALL MRRRR (K, K, G3, K, K, K, MINV, K, K, K, GAM, K)
      CALL WRRRN ('GAMMA', K, K, GAM, K, 0)
C                     ESTIMATION OF THE VARIANCE
      CALL MRRRR (K, K, PHI, K, K, N, TX, K, K, N, V1, K)
      CALL MRRRR (K, K, GAM, K, K, N, XM, K, K, N, V2, K)
      CALL MRRRR (K, K, THETA, K, K, N, TA, K, K, N, V3, K)
      VAR=XD-V1-V2+V3
      CALL TRNRR (K, N, VAR, K, N, K, TVAR, N)
      CALL MRRRR (K, N, VAR, K, N, K, TVAR, N, K, K, VV, K)
      SIGMA=VV/N
      CALL WRRRN ('SIGMA', K, K, SIGMA, K, 0)
C    EIGENVALUES FOR THE NON-CAUSALITY TEST WITH COINTEGRATION FOR
AGGREGATES
      UO(1,1)=1
      UO(2,1)=0
      U(1,1)=0
      U(2,1)=1
      CALL TRNRR (K, 1, UO, K, 1, K, TUO, 1)
      CALL TRNRR (K, 1, U, K, 1, K, TUU, 1)
      CALL MRRRR (K, K, SFP0, K, K, 1, U, K, K, 1, SKB, K)
      CALL TRNRR (K, 1, SKB, K, 1, K, SBK, 1)
      CALL MRRRR (1, K, TUU, 1, K, K, SF00, K, 1, K, S001, 1)
      CALL MRRRR (1, K, S001, 1, K, 1, U, K, 1, 1, SBB, 1)
```

```
      CALL MRRRR (K, 1, SKB, K, 1, K, SBK, 1, K, K, SKK1, K)
      SKKB=TXX-SKK1/SBB
      CALL LINRG (K, SKKB, K, ISKKB, K)
      CALL MRRRR (1, K, TUO, 1, K, K, SF00, K, 1, K, SAA1, 1)
      CALL MRRRR (1, K, SAA1, 1, K, 1, U0, K, 1, 1, SAA, 1)
      CALL MRRRR (1, K, SAA1, 1, K, 1, U, K, 1, 1, SAB, 1)
      SAAB=SAA-SAB**2/SBB
      CALL MRRRR (1, K, TUO, 1, K, K,SF0P, K, 1, K, SAK, 1)
      CALL MRRRR (1, K, TUU, 1, K, K, SF0P, K, 1, K, SBK, 1)
      CALL MRRRR (1, 1, SAB, 1, 1, K, SBK, 1, 1, K, SAK3, 1)
      SAK4=SAK-SAK3/SBB
      CALL TRNRR (1, K, SAK4, 1, K, 1, SAK5, K)
      CALL MRRRR (K, 1, SAK5, K, 1, K, SAK4, 1, K, K, SAK6, K)
      SAK7=SAK6/SAAB
      CALL MRRRR (K, K, SAK7, K, K, K, ISKKB, K, K, K, SAK8, K)
      CALL EVCRG (K, SAK8, K, EVALAR, EVECAR, K)
      CALL WRCRN ('EVAL_AGGREGATE', 1, K, EVALAR, 1, 0)
      CALL WRCRN ('EVEC_AGGREGATE', K, K, EVECAR, K, 0)
C
      TRACE_AGGR=-N*(LOG(1-EVALAR(1))+LOG(1-EVALAR(2)))
      TRACE1_AGGR=-N*(LOG(1-EVALAR(2)))
C                         FIND THE TEST STATICTIC FOR NON-CAUSALITY
      TESTAT_H0=N*LOG((1-EVALAR(1))/(1-EVALA(1)))
C
      WRITE(*,*) "TESTAT_H0:", TESTAT_H0
C                  FIND THE P-VALUE OF THE TEST
      CALL UMACH (2, NOUT)
      DF   = 1.0
```

```
P1= 1-CHIDF(TESTAT_H0,DF)

WRITE(*,*) P1

CLOSE(NOUT)

CLOSE(NIN)

STOP

END
```

2. Simulation Program in order to Test Non-Causality in Cointegrated Systems for Aggregation: The output gives the percentage of rejection for the null hypothesis that is first variable does not cause the second one. To be able to test second variable does not cause the first one, user need to change U matrix as UO and UO matrix as U. The code is the following:

```
***********************************************************************
*           FORTRAN SIMULATION PROGRAM IN ORDER TO TEST               *
*        NON-CAUSALITY IN COINTEGRATED SYSTEMS FOR AGGREGATES         *
***********************************************************************
  USE Numerical_Libraries
*     .. Parameters ..
  PARAMETER (NO=1300,MI=2 NE=NO+MI, NN=NO-100, NOM=NN/MI, NM1=NOM+MI)
*     .. Local Scalars ..
  INTEGER K, N, NM, LIM, COUNT, ISEED, ICOUNT
*     .. Local Arrays ..
  REAL Z(NO,2),X(NN,2),E(NE,2),COV(2,2),,RSIG(2,2),PHI(2,2),
&      THETA(2,2),A(NM1,2),XA(NOM,2),YA(NM1,2),DY(NOM,2),
&     YD(2,NOM),DDY(2,2),TDY(2,2),TYD(2,2),TY(2,NOM),
&     TYY(2,2),TA(2,NOM),YDA(2,2),YAD(2,2),TAA(2,2),
&     ITAA(2,2),A1(2,2),S00(2,2),Z0(2,2),IZ(2,2),TAY(2,2),
&     TYA(2,2),A2(2,2),S0P(2,2),SP0(2,2),A3(2,2),SPP(2,2),
&      Z1(2,2),Z2(2,2),Z3(2,2),Z4(2,2),SA1(2,2),U(2,1),
&      UO(2,1),TUU(1,2),TUO(1,2),Y1(1,2),Y2,Y3(2,1),Y4(1,2),
&     Y5(2,2),Y6(2,2),IY(2,2),Y7(2,1),Y8(1,2),Y9(2,2),
&     Y10(2,2),Y11(2,2),Y12(2,2),Y13(2,2),T1(1,2),T2(2,1),
&     T3(2,2),T4(2,2),T5(2,2),W1(1,2),W2,TESTAT,ALPHAO
  COMPLEX    EVALA(2), EVECA(2,2),EVALAR(2), EVECAR(2,2)
*     .. Executable Statements ..
  K=2
  N=1300
*                   SET THE AR PARAMETER
  PHI(1,1)=1.0
  PHI(1,2)=0.0
```

```
      PHI(2,1)=0.6
      PHI(2,2)=0.0
C                   SET THE ERROR COVARIANCE MATRIX
      COV(1,1) = 2
      COV(1,2) = 0.5
      COV(2,1) = 0.5
      COV(2,2) = 1.5
C                     SET U AND U_orthogonal VECTORS
      U(1,1)=0
      U(2,1)=1
      UO(1,1)=1
      UO(2,1)=0
C
      ICOUNT=0
*                      SET THE NUMBER OF ITERATIONS
      LIM=10000
C                      GENERATION OF ERROR TERMS
      DO 1 L=1,LIM
        IF (L.GT.1) THEN
           N=N+100
        ENDIF
C                    OBTAIN THE CHOLESKY FACTORIZATION
      CALL CHFAC (K, COV, 2, 0.00001, IRANK, RSIG, K)
C                    INITIALIZE SEED OF RANDOM NUMBER GENERATOR
       CALL SYSTEM_CLOCK(COUNT)
       ISEED=COUNT
       CALL RNSET (ISEED)
       CALL RNMVN (N, K, RSIG, K, E, N)
C                     DERIVATION OF TIME SERIES VECTOR Z
      Z(1,1)=E(1,1)
      Z(1,2)=E(1,2)
       DO 10 I=2,N
          Z(I,1)=PHI(1,1)*Z(I-1,1)+PHI(1,2)*Z(I-1,2)+E(I,1)
          Z(I,2)=PHI(2,1)*Z(I-1,1)+PHI(2,2)*Z(I-1,2)+E(I,2)
10 CONTINUE
C                  ELIMINATE THE EFFECT OF INITIAL OBSERVATION
      N=N-100
       DO 20 I=1,N
          X(I,1)=Z(I+100,1)
          X(I,2)=Z(I+100,2)
20 CONTINUE
C                  SAMPLE SIZE FOR THE AGGREGATES
      NM=N/M
C                  THETA VALUES
      IF (M.EQ.2) THEN
          TE=0.151
       ELSE IF (M.EQ.3) THEN
          TE=0.1917
       ELSE IF (M.EQ.5) THEN
           TE=0.215
       ELSE IF (M.EQ.6) THEN
            TE=0.22
       ELSE IF (M.EQ.10) THEN
             TE=0.225
        ENDIF
```

```
      THETA(1,1)=TE
      THETA(1,2)=0
      THETA(2,1)=0
      THETA(2,2)=TE
C     OBTAIN THE THEORETICAL ERRORS FOR THE AGGREGATES FOR M=2,3,5,6,10
      IF (M.EQ.2) THEN
          DO 60 II=1,NM
                IF ((II-1).LT.1) THEN
                      A(II-1,1)=0
                      A(II-1,2)=0
                ENDIF
                DO 50 LJ=M,(2*M-1)
                    IF ((II*M-LJ).LT.1) THEN
                            E(II*M-LJ,1)=0
                            E(II*M-LJ,2)=0
                      ENDIF
50              CONTINUE
                A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &                  (2+PHI(1,1))*E(II*M-1,1)+
     &                  (2+PHI(1,2))*E(II*M-1,2)+
     &                  (1+2*PHI(1,1))*E(II*M-2,1)+
     &                  (1+2*PHI(1,2))*E(II*M-2,2)+
     &                   PHI(1,1)*E(II*M-3,1)+
     &                   PHI(1,2)*E(II*M-3,2)
                 A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &                   (2+PHI(2,1))*E(II*M-1,1)+
     &                   (2+PHI(2,2))*E(II*M-1,2)+
     &                   (1+2*PHI(2,1))*E(II*M-2,1)+
     &                   (1+2*PHI(2,2))*E(II*M-2,2)+
     &                 PHI(2,1)*E(II*M-3,1)+
     &                    PHI(2,2)*E(II*M-3,2)
60 CONTINUE
       ELSE IF (M.EQ.3) THEN
                DO 80 II=1,NM
                      IF ((II-1).LT.1) THEN
                            A(II-1,1)=0
                            A(II-1,2)=0
                      ENDIF
                      DO 70 LJ=M,(2*M-1)
                            IF ((II*M-LJ).LT.1) THEN
                                  E(II*M-LJ,1)=0
                                  E(II*M-LJ,2)=0
                            ENDIF
70                    CONTINUE
                A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &                  (2+PHI(1,1))*E(II*M-1,1)+
     &                  (2+PHI(1,2))*E(II*M-1,2)+
     &                  (3+2*PHI(1,1))*E(II*M-2,1)+
     &                  (3+2*PHI(1,2))*E(II*M-2,2)+
     &                  (2+3*PHI(1,1))*E(II*M-3,1)+
     &                  (2+3*PHI(1,2))*E(II*M-3,2)+
     &                  (1+2*PHI(1,1))*E(II*M-4,1)+
     &                  (1+2*PHI(1,2))*E(II*M-4,2)+
     &                   PHI(1,1)*E(II*M-5,1)+
     &                   PHI(1,2)*E(II*M-5,2)
```

```
          A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &          (2+PHI(2,1))*E(II*M-1,1)+
     &          (2+PHI(2,2))*E(II*M-1,2)+
     &          (3+2*PHI(2,1))*E(II*M-2,1)+
     &          (3+2*PHI(2,2))*E(II*M-2,2)+
     &          (2+3*PHI(2,1))*E(II*M-3,1)+
     &          (2+3*PHI(2,2))*E(II*M-3,2)+
     &          (1+2*PHI(2,1))*E(II*M-4,1)+
     &          (1+2*PHI(2,2))*E(II*M-4,2)+
     &           PHI(2,1)*E(II*M-5,1)+
     &           PHI(2,2)*E(II*M-5,2)
80    CONTINUE
  ELSE IF (M.EQ.5) THEN
      DO 100 II=1,NM
          IF ((II-1).LT.1) THEN
                A(II-1,1)=0
                A(II-1,2)=0
          ENDIF
          DO 90 LJ=M,(2*M-1)
              IF ((II*M-LJ).LT.1) THEN
                      E(II*M-LJ,1)=0
                      E(II*M-LJ,2)=0
                ENDIF
90        CONTINUE
          A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &          (2+PHI(1,1))*E(II*M-1,1)+
     &          (2+PHI(1,2))*E(II*M-1,2)+
     &          (3+2*PHI(1,1))*E(II*M-2,1)+
     &          (3+2*PHI(1,2))*E(II*M-2,2)+
     &          (4+3*PHI(1,1))*E(II*M-3,1)+
     &          (4+3*PHI(1,2))*E(II*M-3,2)+
     &          (5+4*PHI(1,1))*E(II*M-4,1)+
     &          (5+4*PHI(1,2))*E(II*M-4,2)+
     &          (4+5*PHI(1,1))*E(II*M-5,1)+
     &          (4+5*PHI(1,2))*E(II*M-5,2)+
     &          (3+4*PHI(1,1))*E(II*M-6,1)+
     &          (3+4*PHI(1,2))*E(II*M-6,2)+
     &          (2+3*PHI(1,1))*E(II*M-7,1)+
     &          (2+3*PHI(1,2))*E(II*M-7,2)+
     &          (1+2*PHI(1,1))*E(II*M-8,1)+
     &          (1+2*PHI(1,2))*E(II*M-8,2)+
     &           PHI(1,1)*E(II*M-9,1)+
     &           PHI(1,2)*E(II*M-9,2)
          A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &          (2+PHI(2,1))*E(II*M-1,1)+
     &          (2+PHI(2,2))*E(II*M-1,2)+
     &          (3+2*PHI(2,1))*E(II*M-2,1)+
     &          (3+2*PHI(2,2))*E(II*M-2,2)+
     &          (4+3*PHI(2,1))*E(II*M-3,1)+
     &          (4+3*PHI(2,2))*E(II*M-3,2)+
     &          (5+4*PHI(2,1))*E(II*M-4,1)+
     &          (5+4*PHI(2,2))*E(II*M-4,2)+
     &          (4+5*PHI(2,1))*E(II*M-5,1)+
     &          (4+5*PHI(2,2))*E(II*M-5,2)+
     &          (3+4*PHI(2,1))*E(II*M-6,1)+
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     &                (3+4*PHI(2,2))*E(II*M-6,2)+
     &                (2+3*PHI(2,1))*E(II*M-7,1)+
     &                (2+3*PHI(2,2))*E(II*M-7,2)+
     &                (1+2*PHI(2,1))*E(II*M-8,1)+
     &                (1+2*PHI(2,2))*E(II*M-8,2)+
     &                 PHI(2,1)*E(II*M-9,1)+
     &                 PHI(2,2)*E(II*M-9,2)
100         CONTINUE
   ELSE IF (M.EQ.6) THEN
      DO 120 II=1,NM
            IF ((II-1).LT.1) THEN
                  A(II-1,1)=0
                  A(II-1,2)=0
            ENDIF
            DO 110 LJ=M,(2*M-1)
                  IF ((II*M-LJ).LT.1) THEN
                        E(II*M-LJ,1)=0
                        E(II*M-LJ,2)=0
                  ENDIF
110          CONTINUE
            A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &                (2+PHI(1,1))*E(II*M-1,1)+
     &                (2+PHI(1,2))*E(II*M-1,2)+
     &                (3+2*PHI(1,1))*E(II*M-2,1)+
     &                (3+2*PHI(1,2))*E(II*M-2,2)+
     &                (4+3*PHI(1,1))*E(II*M-3,1)+
     &                (4+3*PHI(1,2))*E(II*M-3,2)+
     &                (5+4*PHI(1,1))*E(II*M-4,1)+
     &                (5+4*PHI(1,2))*E(II*M-4,2)+
     &                (6+5*PHI(1,1))*E(II*M-5,1)+
     &                (6+5*PHI(1,2))*E(II*M-5,2)+
     &                (5+6*PHI(1,1))*E(II*M-6,1)+
     &                (5+6*PHI(1,2))*E(II*M-6,2)+
     &                (4+5*PHI(1,1))*E(II*M-7,1)+
     &                (4+5*PHI(1,2))*E(II*M-7,2)+
     &                (3+4*PHI(1,1))*E(II*M-8,1)+
     &                (3+4*PHI(1,2))*E(II*M-8,2)+
     &                (2+3*PHI(1,1))*E(II*M-9,1)+
     &                (2+3*PHI(1,2))*E(II*M-9,2)+
     &                (1+2*PHI(1,1))*E(II*M-10,1)+
     &                (1+2*PHI(1,2))*E(II*M-10,2)+
     &                 PHI(1,1)*E(II*M-11,1)+
     &                 PHI(1,2)*E(II*M-11,2)
            A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &                (2+PHI(1,1))*E(II*M-1,1)+
     &                (2+PHI(1,2))*E(II*M-1,2)+
     &                (3+2*PHI(1,1))*E(II*M-2,1)+
     &                (3+2*PHI(1,2))*E(II*M-2,2)+
     &                (4+3*PHI(1,1))*E(II*M-3,1)+
     &                (4+3*PHI(1,2))*E(II*M-3,2)+
     &                (5+4*PHI(1,1))*E(II*M-4,1)+
     &                (5+4*PHI(1,2))*E(II*M-4,2)+
     &                (6+5*PHI(1,1))*E(II*M-5,1)+
     &                (6+5*PHI(1,2))*E(II*M-5,2)+
     &                (5+6*PHI(1,1))*E(II*M-6,1)+
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     &                (5+6*PHI(1,2))*E(II*M-6,2)+
     &                (4+5*PHI(1,1))*E(II*M-7,1)+
     &                (4+5*PHI(1,2))*E(II*M-7,2)+
     &                (3+4*PHI(1,1))*E(II*M-8,1)+
     &                (3+4*PHI(1,2))*E(II*M-8,2)+
     &                (2+3*PHI(1,1))*E(II*M-9,1)+
     &                (2+3*PHI(1,2))*E(II*M-9,2)+
     &                (1+2*PHI(1,1))*E(II*M-10,1)+
     &                (1+2*PHI(1,2))*E(II*M-10,2)+
     &                 PHI(1,1)*E(II*M-11,1)+
     &                 PHI(1,2)*E(II*M-11,2)
120          CONTINUE
      ELSE IF (M.EQ.10) THEN
             DO 140 II=1,NM
                   IF ((II-1).LT.1) THEN
                         A(II-1,1)=0
                         A(II-1,2)=0
                   ENDIF
                   DO 130 LJ=M,(2*M-1)
                       IF ((II*M-LJ).LT.1) THEN
                               E(II*M-LJ,1)=0
                               E(II*M-LJ,2)=0
                         ENDIF
130                CONTINUE
             A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &                (2+PHI(1,1))*E(II*M-1,1)+
     &                (2+PHI(1,2))*E(II*M-1,2)+
     &                (3+2*PHI(1,1))*E(II*M-2,1)+
     &                (3+2*PHI(1,2))*E(II*M-2,2)+
     &                (4+3*PHI(1,1))*E(II*M-3,1)+
     &                (4+3*PHI(1,2))*E(II*M-3,2)+
     &                (5+4*PHI(1,1))*E(II*M-4,1)+
     &                (5+4*PHI(1,2))*E(II*M-4,2)+
     &                (6+5*PHI(1,1))*E(II*M-5,1)+
     &                (6+5*PHI(1,2))*E(II*M-5,2)+
     &                (7+6*PHI(1,1))*E(II*M-6,1)+
     &                (7+6*PHI(1,2))*E(II*M-6,2)+
     &                (8+7*PHI(1,1))*E(II*M-7,1)+
     &                (8+7*PHI(1,2))*E(II*M-7,2)+
     &                (9+8*PHI(1,1))*E(II*M-8,1)+
     &                (9+8*PHI(1,2))*E(II*M-8,2)+
     &                (10+9*PHI(1,1))*E(II*M-9,1)+
     &                (10+9*PHI(1,2))*E(II*M-9,2)+
     &                (9+10*PHI(1,1))*E(II*M-10,1)+
     &                (9+10*PHI(1,2))*E(II*M-10,2)+
     &                (8+9*PHI(1,1))*E(II*M-11,1)+
     &                (8+9*PHI(1,2))*E(II*M-11,2)+
     &                (7+8*PHI(1,1))*E(II*M-12,1)+
     &                (7+8*PHI(1,2))*E(II*M-12,2)+
     &                (6+7*PHI(1,1))*E(II*M-13,1)+
     &                (6+7*PHI(1,2))*E(II*M-13,2)+
     &                (5+6*PHI(1,1))*E(II*M-14,1)+
     &                (5+6*PHI(1,2))*E(II*M-14,2)+
     &                (4+5*PHI(1,1))*E(II*M-15,1)+
     &                (4+5*PHI(1,2))*E(II*M-15,2)+
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     &              (3+4*PHI(1,1))*E(II*M-16,1)+
     &              (3+4*PHI(1,2))*E(II*M-16,2)+
     &              (2+3*PHI(1,1))*E(II*M-17,1)+
     &              (2+3*PHI(1,2))*E(II*M-17,2)+
     &              (1+2*PHI(1,1))*E(II*M-18,1)+
     &              (1+2*PHI(1,2))*E(II*M-18,2)+
     &               PHI(1,1)*E(II*M-19,1)+
     &               PHI(1,2)*E(II*M-19,2)
          A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &              (2+PHI(2,1))*E(II*M-1,1)+
     &              (2+PHI(2,2))*E(II*M-1,2)+
     &              (3+2*PHI(2,1))*E(II*M-2,1)+
     &              (3+2*PHI(2,2))*E(II*M-2,2)+
     &              (4+3*PHI(2,1))*E(II*M-3,1)+
     &              (4+3*PHI(2,2))*E(II*M-3,2)+
     &              (5+4*PHI(2,1))*E(II*M-4,1)+
     &              (5+4*PHI(2,2))*E(II*M-4,2)+
     &              (6+5*PHI(2,1))*E(II*M-5,1)+
     &              (6+5*PHI(2,2))*E(II*M-5,2)+
     &              (7+6*PHI(2,1))*E(II*M-6,1)+
     &              (7+6*PHI(2,2))*E(II*M-6,2)+
     &              (8+7*PHI(2,1))*E(II*M-7,1)+
     &              (8+7*PHI(2,2))*E(II*M-7,2)+
     &              (9+8*PHI(2,1))*E(II*M-8,1)+
     &              (9+8*PHI(2,2))*E(II*M-8,2)+
     &              (10+9*PHI(2,1))*E(II*M-9,1)+
     &              (10+9*PHI(2,2))*E(II*M-9,2)+
     &              (9+10*PHI(2,1))*E(II*M-10,1)+
     &              (9+10*PHI(2,2))*E(II*M-10,2)+
     &              (8+9*PHI(2,1))*E(II*M-11,1)+
     &              (8+9*PHI(2,2))*E(II*M-11,2)+
     &              (7+8*PHI(2,1))*E(II*M-12,1)+
     &              (7+8*PHI(2,2))*E(II*M-12,2)+
     &              (6+7*PHI(2,1))*E(II*M-13,1)+
     &              (6+7*PHI(2,2))*E(II*M-13,2)+
     &              (5+6*PHI(2,1))*E(II*M-14,1)+
     &              (5+6*PHI(2,2))*E(II*M-14,2)+
     &              (4+5*PHI(2,1))*E(II*M-15,1)+
     &              (4+5*PHI(2,2))*E(II*M-15,2)+
     &              (3+4*PHI(2,1))*E(II*M-16,1)+
     &              (3+4*PHI(2,2))*E(II*M-16,2)+
     &              (2+3*PHI(2,1))*E(II*M-17,1)+
     &              (2+3*PHI(2,2))*E(II*M-17,2)+
     &              (1+2*PHI(2,1))*E(II*M-18,1)+
     &              (1+2*PHI(2,2))*E(II*M-18,2)+
     &               PHI(2,1)*E(II*M-19,1)+
     &               PHI(2,2)*E(II*M-19,2)
140       CONTINUE
      ENDIF
C
          DO 105 II=1,NM
              IF ((II-1).LT.1) THEN
                  A(II-1,1)=0
                  A(II-1,2)=0
              ENDIF
```

```
                DO 95 LJ=M,(2*M-1)
                    IF ((II*M-LJ).LT.1) THEN
                            E(II*M-LJ,1)=0
                            E(II*M-LJ,2)=0
                      ENDIF
95              CONTINUE
          A(II,1)=THETA(1,1)*A(II-1,1)+E(II*M,1)+
     &               (2+PHI(1,1))*E(II*M-1,1)+
     &               (2+PHI(1,2))*E(II*M-1,2)+
     &               (3+2*PHI(1,1))*E(II*M-2,1)+
     &               (3+2*PHI(1,2))*E(II*M-2,2)+
     &               (4+3*PHI(1,1))*E(II*M-3,1)+
     &               (4+3*PHI(1,2))*E(II*M-3,2)+
     &               (5+4*PHI(1,1))*E(II*M-4,1)+
     &               (5+4*PHI(1,2))*E(II*M-4,2)+
     &               (4+5*PHI(1,1))*E(II*M-5,1)+
     &               (4+5*PHI(1,2))*E(II*M-5,2)+
     &               (3+4*PHI(1,1))*E(II*M-6,1)+
     &               (3+4*PHI(1,2))*E(II*M-6,2)+
     &               (2+3*PHI(1,1))*E(II*M-7,1)+
     &               (2+3*PHI(1,2))*E(II*M-7,2)+
     &               (1+2*PHI(1,1))*E(II*M-8,1)+
     &               (1+2*PHI(1,2))*E(II*M-8,2)+
     &                PHI(1,1)*E(II*M-9,1)+
     &                PHI(1,2)*E(II*M-9,2)
          A(II,2)=THETA(2,2)*A(II-1,2)+E(II*M,2)+
     &               (2+PHI(2,1))*E(II*M-1,1)+
     &               (2+PHI(2,2))*E(II*M-1,2)+
     &               (3+2*PHI(2,1))*E(II*M-2,1)+
     &               (3+2*PHI(2,2))*E(II*M-2,2)+
     &               (4+3*PHI(2,1))*E(II*M-3,1)+
     &               (4+3*PHI(2,2))*E(II*M-3,2)+
     &               (5+4*PHI(2,1))*E(II*M-4,1)+
     &               (5+4*PHI(2,2))*E(II*M-4,2)+
     &               (4+5*PHI(2,1))*E(II*M-5,1)+
     &               (4+5*PHI(2,2))*E(II*M-5,2)+
     &               (3+4*PHI(2,1))*E(II*M-6,1)+
     &               (3+4*PHI(2,2))*E(II*M-6,2)+
     &               (2+3*PHI(2,1))*E(II*M-7,1)+
     &               (2+3*PHI(2,2))*E(II*M-7,2)+
     &               (1+2*PHI(2,1))*E(II*M-8,1)+
     &               (1+2*PHI(2,2))*E(II*M-8,2)+
     &                PHI(2,1)*E(II*M-9,1)+
     &                PHI(2,2)*E(II*M-9,2)
105       CONTINUE
C
C                           OBTAIN THE AGGREGATES
      XA(1,1)=0
      XA(1,2)=0
      DO 100 I=1,NM
          DO 110 J=1,M
            XA(I,1)=XA(I,1)+X(I*M-J+1,1)
            XA(I,2)=XA(I,2)+X(I*M-J+1,2)
110      CONTINUE
100     CONTINUE
```

```
C
      YA(1,1)=0
      YA(1,2)=0
      DO 120 I=1,NM
          YA(I+1,1)=XA(I,1)
          YA(I+1,2)=XA(I,2)
120   CONTINUE
C           FIND THE DIFFERENCED SERIES   (1-B)YA(t)
      DY(1,1)=0
      DY(1,2)=0
      DO 130 J=1,NM
        DY(J,1)=YA(J,1)-YA(J-1,1)
        DY(J,2)=YA(J,2)-YA(J-1,2)
130    CONTINUE
      DO 140 IL=1,NM
          XA(IL,1)=YA(IL,1)
          XA(IL,2)=YA(IL,2)
140   CONTINUE
C                 FIND THE EIGENVALUES FOR THE DENOMINATOR
      CALL TRNRR (NM, K, DY, NM, K, NM, YD, K)
      CALL MRRRR (K, NM, YD, K, NM, K, DY, NM, K, K,DDY, K)
      CALL MRRRR (K, NM, YD, K, NM, K, XA, NM, K, K,TDY, K)
      CALL TRNRR (NM, K, XA, NM, K, NM, TY, K)
      CALL MRRRR (K, NM, TY, K, NM, K, XA, NM, K, K,TYY, K)
      CALL TRNRR (NM, K, A, NM, K, NM, TA, K)
      CALL MRRRR (K, NM, YD, K, NM, K, A, NM, K, K,YDA, K)
      CALL TRNRR (K, K, YDA, K, K, K, YAD, K)
      CALL MRRRR (K, NM, TA, K, NM, K, A, NM, K, K,TAA, K)
      CALL LINRG (K, TAA, K, ITAA, K)
      CALL MRRRR (K, K, YDA, K, K, K, ITAA, K, K, K,A1, K)
      CALL MRRRR (K, K, A1, K, K, K, YAD, K, K, K,S00, K)
      Z0=DDY-S00
      CALL LINRG (K, Z0, K, IZ, K)
      CALL MRRRR (K, NM, TA, K, NM, K, XA, NM, K, K,TAY, K)
      CALL TRNRR (K, K, TAY, K, K, K, TYA, K)
      CALL MRRRR (K, K, YDA, K, K, K, ITAA, K, K, K,A2, K)
      CALL MRRRR (K, K, A2, K, K, K, TAY, K, K, K,S0P, K)
      CALL TRNRR (K, K, S0P, K, K, K, SP0, K)
      CALL MRRRR (K, K, TYA, K, K, K, ITAA, K, K, K,A3, K)
      CALL MRRRR (K, K, A3, K, K, K, TAY, K, K, K,SPP, K)
      Z1=TDY-S0P
      CALL TRNRR (K, K, Z1, K, K, K, Z2, K)
      Z3=TYY-SPP
      CALL LINRG (K, Z3, K, Z4, K)
      CALL MRRRR (K, K, Z2, K, K, K, IZ, K, K, K, SA1, K)
      CALL MRRRR (K, K, SA1, K, K, K, Z1, K, K, K, SA2, K)
      CALL MRRRR (K, K, SA2, K, K, K, Z4, K, K, K, SA3, K)
      CALL EVCRG (K, SA3, K, EVALA, EVECA, K)

C                 FIND THE EIGENVALUES FOR THE NUMERATOR
      CALL TRNRR (K, 1, UO, K, 1, K, TUO, 1)
      CALL TRNRR (K, 1, U, K, 1, K, TUU, 1)
      CALL MRRRR (1, K, TUU, 1, K, K, Z0, K, 1, K,Y1, 1)
      CALL MRRRR (1, K, Y1, 1, K, 1, U, K, 1, 1,Y2, 1)
      CALL MRRRR (K, K, Z2, K, K, 1, U, K, K, 1,Y3, K)
```

```
      CALL TRNRR (K, 1, Y3, K, 1, K, Y4, 1)
      CALL MRRRR (K, 1, Y3, K, 1, K, Y4, 1, K, K,Y5, K)
      Y6=TYY-(1/Y2)*Y5
      CALL LINRG (K, Y6, K, IY, K)
      CALL MRRRR (K, K, Z0, K, K, 1, U, K, K, 1,Y7, K)
      CALL TRNRR (K, 1, Y7, K, 1, K, Y8, 1)
      CALL MRRRR (K, 1, Y7, K, 1, K, Y8, 1, K, K,Y9, K)
      Y10=Z0-Y9/Y2
      CALL MRRRR (K, 1, Y7, K, 1, K, Y4, 1, K, K,Y11, K)
      Y12=Z1-Y11/Y2
      CALL TRNRR (K, K, Y12, K, K, K, Y13, K)
      CALL MRRRR (1, K, TUO, 1, K, K, Y12, K, 1, K, T1, 1)
      CALL TRNRR (1, K, T1, 1, K, 1, T2, K)
      CALL MRRRR (K, 1, T2, K, 1, K, T1, 1, K, K, T3, K)
      CALL MRRRR (1, K, TUO, 1, K, K, Z0, K, 1, K, W1, 1)
      CALL MRRRR (1, K, W1, 1, K, 1, UO, K, 1, 1, W2, 1)
      T4=T3/W2
      CALL MRRRR (K, K, T4, K, K, K, IY, K, K, K, T5, K)
      CALL EVCRG (K, T5, K, EVALAR, EVECAR, K)
C     FIND THE TEST STATISTIC FOR NON-CAUSALITY WITH COINTEGRATION FOR
AGGREGATES
      TESTAT=NM*LOG((1-EVALAR(1))/(1-EVALA(1)))
C           FIND THE CHI-SQUARE VALUE FOR ALPHA=0.05 AND DF=1
      CALL UMACH (2, NOUT)
      P  = 0.95
      DF = 1.0
      CHI  = CHIIN(P,DF)
C           COUNT THE REJECTED HYPOTHESIS OF NON-CAUSALITY
      IF (TESTAT.GE.CHI) THEN
            ICOUNT=ICOUNT+1
      ELSE
            ICOUNT=ICOUNT
      ENDIF
C
1     CONTINUE
C           FIND THE PERCENT REJECTION OF NULL HYPOTHESIS
      ALPHA0=1.0*ICOUNT/LIM
      WRITE(*,*) ICOUNT
      WRITE(*,*) ALPHA0
      STOP
      END
```